Correction of exam 2014/2015

Exercise 1 *We recall the ODE*

$$(t^2+1)\hat{y}'(t) = t\hat{y}(t)^2 - t. \tag{1}$$

1. *We have*

$$\begin{aligned}(t^2+1)\hat{y}'(t) = t\hat{y}(t)^2 - t &\Leftrightarrow (t^2+1)\hat{y}'(t) = t\big(\hat{y}(t)^2-1\big)\\ &\Leftrightarrow (t^2+1)\hat{y}'(t) = t\big(\hat{y}(t)-1\big)\big(\hat{y}(t)+1\big).\end{aligned}$$

   Since a constant solution $\hat{y}$ satisfies $\hat{y}'=0$, we obtain that $t\mapsto -1$ ***and*** *$t\mapsto 1$* ***are two constant solutions of*** (1).

2. *To prove the existence of the solution, we write the ODE* (1) *on the form*

$$\hat{y}'(t) = f(t,\hat{y}(t))$$

   for some function f. Since for all $t\in\mathbb{R}$, $t^2+1>0$, we have

$$\begin{aligned}(t^2+1)\hat{y}'(t) = t\hat{y}(t)^2 - t &\Leftrightarrow \hat{y}'(t) = \frac{t}{t^2+1}\big(\hat{y}(t)^2-1\big)\\ &\Leftrightarrow \hat{y}'(t) = \frac{t}{t^2+1}\hat{y}(t)^2 - \frac{t}{t^2+1}\\ &\Leftrightarrow \hat{y}'(t) = f(t,\hat{y}(t))\end{aligned}$$

   where

$$f(t,y) = \frac{t}{t^2+1}y^2 - \frac{t}{t^2+1}.$$

   Since $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is of class $\mathscr{C}^1$ with respect to (t,y), the Cauchy-Lipschitz theorem gives us ***the existence of a unique maximal solution*** *$\hat{y}\in\mathscr{C}^1(J)$* ***where*** *$J\subset\mathbb{R}$* ***satisfying*** *$0\in J$* ***and*** *$\hat{y}(0)=0$.*

3. *We now prove that $J=\mathbb{R}$. By using Proposition 2.3 of the lecture notes, it is sufficient to prove that $\hat{y}$ is bounded on J. We will prove that for all $t\in J$, $\hat{y}(t)\in]-1,1[$ by contradiction.*
   Assume that there there exists $t_0\in J$ such that $\hat{y}(t_0)\notin]-1,1[$. There are two cases : $\hat{y}(t_0)\geq 1$ or $\hat{y}(t_0)\leq 1$.
   We begin with the case $\hat{y}(t_0)\geq 1$.
   If $\hat{y}(t_0)\geq 1$. Since $\hat{y}(0)=0$, $\hat{y}(t_0)>1$ and $\hat{y}$ is continious on J, the intermediate values theorem (T.V.I.) gives us that there exists $t_1\in J$ between 0 and t_0 such that

$$\hat{y}(t_1) = 1.$$

   Then, $\hat{y}$ is the maximal solution of (1) *such that $\hat{y}(t_1)=1$. However, we prooved in Question 1) that $t\mapsto 1$ is a constant solution of* (1) *defined on $\mathbb{R}$. Its value on t_1 is also 1. The unicity of the maximal solution (Cauchy Lipschitz theorem) implies that*

$$\forall t\in\mathbb{R},\quad \hat{y}(t) = 1$$

   which contracdicts the fact the $\hat{y}(0)=0$. Then, we obtain that for all $t\in J$, $\hat{y}(t)<1$.
   We use the same method for the case $\hat{y}(t_0)\leq -1$, we obtain that for all $t\in J$, $\hat{y}(t)>-1$.
   Then, for all $t\in J$, $\hat{y}(t)\in]-1;1[$ and the maximal solution $\hat{y}$ is bounded over the interval J. ***We deduce from Proposition 2.3 of the lecture notes that*** *$J=\mathbb{R}$.*

4. *By setting $\hat{z} = \hat{y} - 1$, we have $\hat{z}' = \hat{y}'$. We can write* (1) *as*

$$\begin{aligned}(t^2+1)\hat{y}'(t) = t\hat{y}(t)^2 - t &\Leftrightarrow (t^2+1)\hat{z}'(t) = t(\hat{z}(t)+1)^2 - t\\ &\Leftrightarrow (t^2+1)\hat{z}'(t) = t\left(\hat{z}(t)^2 + 2\hat{z}(t) + 1\right) - t\\ &\Leftrightarrow (t^2+1)\hat{z}'(t) = t\left(\hat{z}(t)^2 + 2\hat{z}(t)\right).\end{aligned}$$

5. *By setting $\hat{w} = \frac{1}{\hat{z}}$, we have $\hat{z}' = -\frac{\hat{w}'}{\hat{w}^2}$. Since for all $t \in \mathbb{R}$, $\hat{y}(t) \in]-1,1[$, the function $\hat{z} = \hat{y} - 1$ can not vanish and the function $\hat{w}$ is well defined on $\mathbb{R}$. Then, we have*

$$\begin{aligned}(t^2+1)\hat{z}'(t) = t\left(\hat{z}(t)^2 + 2\hat{z}(t)\right) &\Leftrightarrow (t^2+1)\left(-\frac{\hat{w}'(t)}{\hat{w}(t)^2}\right) = t\left(\left(\frac{1}{\hat{w}(t)}\right)^2 + 2\frac{1}{\hat{w}(t)}\right)\\ &\Leftrightarrow -(t^2+1)\hat{w}'(t) = t(1+2\hat{w}(t))\\ &\Leftrightarrow \hat{w}'(t) = -\frac{t}{t^2+1} - \frac{2t}{t^2+1}\hat{w}(t)\\ &\Leftrightarrow \hat{w}'(t) = -\frac{2t}{t^2+1}\hat{w}(t) - \frac{t}{t^2+1}.\end{aligned}$$

6. *The general solution $\hat{w}$ of the ODE*

$$\hat{w}'(t) = -\frac{2t}{t^2+1}\hat{w}(t) - \frac{t}{t^2+1} \tag{2}$$

is on the form

$$\hat{w} = \hat{w}_1 + \hat{w}_2$$

where $\hat{w}_1$ is the general solution of the homogenous equation

$$\hat{w}'(t) = -\frac{2t}{t^2+1}\hat{w}(t)$$

and $\hat{w}_2$ is a particular solution of (2)*. We firstly look for the general solution $\hat{w}_1$ of the homogenous equation. The general solution is on the form*

$$\hat{w}_1(t) = C\exp(A(t))$$

where A is a primitive function of $t \mapsto -\frac{2t}{t^2+1}$ and $C \in \mathbb{R}$. Since $\frac{2t}{t^2+1}$ is on the form $\frac{u'}{u}$, we obtain that $t \mapsto -\ln(t^2+1)$ is a primitive function of $t \mapsto -\frac{2t}{t^2+1}$ defined on $\mathbb{R}$. Then, the general solution of the homogenous equation is

$$\hat{w}_1(t) = C\exp\left(-\ln\left(t^2+1\right)\right) = \frac{C}{t^2+1}, \quad C \in \mathbb{R}.$$

We now look for a particular solution of (2)*. We directly see that the constant function $\hat{w}_1 = -\frac{1}{2}$ is a constant particular solution solution of* (2)*. We could also found a particular solution with the variation of parameters method.*
Then, the general solution of the ODE (2) *is*

$$\hat{w}(t) = \hat{w}_1(t) + \hat{w}_2(t) = \frac{C}{t^2+1} - \frac{1}{2}, \quad C \in \mathbb{R}.$$

The initial condition $\hat{w}(0) = -1$ implies that $C = -\frac{1}{2}$. Then, the solution of (2) *satisfying $\hat{w}(0) = -1$ is*
$\boxed{\hat{w}(t) = -\frac{1}{2}\left(\frac{1}{t^2+1} + 1\right) = -\frac{t^2+2}{2(t^2+1)}}$

7. *Since* $\hat{y}(0) = 0 \Leftrightarrow \hat{w}(0) = -1$, *the solution* $\hat{y}$ *of ODE* (1) *with intial condition* $\hat{y}(0) = 0$ *is*

$$\begin{aligned}\hat{y}(t) &= \hat{z}(t) + 1 = \frac{1}{\hat{w}(t)} + 1 \\ &= -\frac{2(t^2+1)}{t^2+2} + 1 = -\frac{t^2}{t^2+2}.\end{aligned}$$

8. *A numerical scheme is a method whose purpose is to compute* y_n *which is an approximation of* $\hat{y}(t^n)$. *The explicit Euler scheme is given by*

$$y_{n+1} = y_n + \Delta t\, f(t_n, y_n).$$

In this particular case, we have

$$f(t,y) = \frac{t}{t^2+1} y^2 - \frac{t}{t^2+1}.$$

Then, the explicit Euler scheme is given by $\boxed{y_{n+1} = y_n + \Delta t \left(\frac{t_n}{t_n^2+1} y_n^2 - \frac{t_n}{t_n^2+1}\right).}$

9. *The implicit Euler scheme is given by*

$$y_{n+1} = y_n + \Delta t\, f(t_{n+1}, y_{n+1}),$$

where

$$f(t,y) = \frac{t}{t^2+1} y^2 - \frac{t}{t^2+1}.$$

Then, the implicit Euler scheme is given by $\boxed{y_{n+1} = y_n + \Delta t \left(\frac{t_{n+1}}{t_{n+1}^2+1} y_{n+1}^2 - \frac{t_{n+1}}{t_{n+1}^2+1}\right).}$

Exercise 2 *We aim at solving the autonomous (*i.e. f *does not depend on* t*) ordinary differential equation*

$$\begin{cases} \hat{y}'(t) = f\big(\hat{y}(t)\big), \\ \hat{y}(0) = \frac{1}{2}. \end{cases} \tag{3}$$

We assume that f *is of class* $\mathscr{C}^2(\mathbb{R})$.

1. *Since* $f : \mathbb{R} \to \mathbb{R}$ *is of class* $\mathscr{C}^1$ *with respect to* y, *the Cauchy-Lipschitz theorem gives us* ***the existence of a unique maximal solution*** $\hat{y} \in \mathscr{C}^1(J)$ ***where*** $J \subset \mathbb{R}$ ***satisfying*** $0 \in J$ ***and*** $\hat{y}(0) = \frac{1}{2}$.

   For the regularity of the solution $\hat{y}$, *we use the Proposition 2.1 p. 24 of the lecture notes. Since* f *is of class* $\mathscr{C}^2$, *the maximal solution* $\hat{y}$ *is of class* $\mathscr{C}^3$ *on his intervall definition* $J \subset \mathbb{R}$.

2. (a) *If* $f(y) = 1$, *the ODE* (3) *becomes*

   $$\hat{y}'(t) = 1.$$

   The general solution of this ODE is

   $$\hat{y}(t) = t + C, \quad C \in \mathbb{R}.$$

   Since $\hat{y}(0) = \frac{1}{2}$, *we have* $C = \frac{1}{2}$. *Then, the solution of the ODE with initial condition* $\hat{y}(0) = \frac{1}{2}$ *is* $\boxed{\hat{y}(t) = t + \frac{1}{2}.}$

(b) *If $f(y) = 10y$, the ODE* (3) *becomes*

$$\hat{y}'(t) = 10\,y(t).$$

The general solution of this ODE is

$$\hat{y}(t) = C\,e^{10\,t}, \quad C \in \mathbb{R}.$$

Since $\hat{y}(0) = \frac{1}{2}$, we have $C = \frac{1}{2}$. Then, the solution of the ODE with initial condition $\hat{y}(0) = \frac{1}{2}$ is $\boxed{\hat{y}(t) = \frac{1}{2}e^{10\,t}.}$

3. *In the general case, we cannot provide an explicit expression. That is why we aim at constructing approximate values of the solution at some points. More precisely, we fixe some integer $N \geq 2$ and we set*

$$\forall n \in [\![1, N]\!],\ t^n = (n-1)\Delta t \quad \textit{where} \quad \Delta t = \frac{3}{N-1}.$$

A numerical scheme is a method whose purpose is to compute y_n which is an approximation of $\hat{y}(t^n)$.

(a) *In the general case, we have* $\boxed{y_n \neq \hat{y}(t^n).}$ *But we have to bear in mind that* $\boxed{y_n \approx \hat{y}(t^n).}$

(b) *We have*

$$t^1 = (1-1)\Delta t = 0,$$
$$t^2 = (2-1)\Delta t = \Delta t,$$
$$t^N = (N-1)\Delta t = (N-1)\frac{3}{N-1} = 3.$$

(c) i. *In this case, the Euler explicit scheme can be written as* $\boxed{y_{n+1} = y_n + \Delta t\, f(y_n).}$

ii. ***We need only one value to initialize the sequence*** *because it is a one step scheme.*

iii. *The algorithm is the following:*

Algorithm 1 Resolution of ODE (3) with a Euler explicit scheme

Data: N
$\Delta x = \frac{3}{N-1}$
y=[];
y(1)=1/2;
for i from 1 to N-1
$y(i+1) = y(i) + \Delta t\, f(y(i))$;
end

iv. ***The order of the explicit Euler scheme is one.***

(d) *We are interesting in the multi-step scheme*

$$z_{n+3} - z_{n+1} = \Delta t\left(\frac{7}{3}f(z_{n+2}) - \frac{2}{3}f(z_{n+1}) + \frac{1}{3}f(z_n)\right). \tag{4}$$

i. *Scheme* (4) *is a 3-step scheme (see Definition 2.9 in the lecture notes) characterized by*

$$\alpha_0 = 0, \quad \alpha_1 = -1, \quad \alpha_2 = 0, \quad \alpha_3 = 1$$
$$\beta_0 = \tfrac{1}{3}, \quad \beta_1 = -\tfrac{2}{3}, \quad \beta_2 = \tfrac{7}{3}, \quad \beta_3 = 0.$$

According to Prop. 2.6, this scheme is consistant if and only if

$$\sum_{k=0}^{3} \alpha_k = 0 \textit{ and } \sum_{k=0}^{3} k\alpha_k = \sum_{k=0}^{3} \beta_k.$$

We easily check that

$$\sum_{k=0}^{3} \alpha_k = 0 - 1 + 0 + 1 = 0$$

$$\sum_{k=0}^{3} k\,\alpha_k = 0 \times 0 + 1 \times (-1) + 2 \times 0 + 3 \times 1 = 2,$$

$$\sum_{k=0}^{3} \beta_k = \frac{1}{3} - \frac{2}{3} + \frac{7}{3} = \frac{6}{3} = 2 = \sum_{k=0}^{3} k\,\alpha_k.$$

*Then, **the scheme is consistant**.*

ii. *The Dahlquist statement (Prop. 2.7) ensures that the Scheme* (4) *is stable if the polynomial* $\rho(x) = \sum_{0\le k\le 3} \alpha_k x^k$ *has only roots of modulo less than* 1 *and if its roots of modulus* 1 *are simple. Here, we have*

$$\rho(x) = x^4 - x^2 = x^2\left(x^2 - 1\right) = x^2(x-1)(x+1)$$

whose roots are -1, 0 *and* 1 *where* -1 *and* 1 *are single roots. Then, **Scheme** (4) **is stable**.*

iii. *Stability and consistency imply the **convergence of the scheme** due to the Lax–Richtmyer theorem (Th. 2.2). The fact that a scheme is convergent means that if we denote by* (y_n) *the numerical solution and by* $\hat{y}$ *the exact solution, then* $\hat{y}(t^n) - y_n$ *goes to* 0 *as* Δt *goes to* 0*: the more* Δt *decreases, the more accurate the numerical solution (close to the exact solution). The scheme is convergent if*

$$\lim_{\Delta t\to 0} \max_{1\le n\le N} \left\| y_n - \hat{y}(t^n) \right\| = 0$$

iv. *We can write Scheme* (4) *as*

$$z_{n+3} = z_{n+1} + \Delta t\left(\frac{7}{3} f(z_{n+2}) - \frac{2}{3} f(z_{n+1}) + \frac{1}{3} f(z_n)\right).$$

Thus, given z_n, z_{n+1} *and* z_{n+2} *we can directly compute* z_{n+3}*. **The scheme is explicit**.*

v. *We apply Prop. 2.8 and assess each i* ≥ 2 *until the relation in Prop. 2.8 is not satisfied:*

- $i = 2$: $\sum_{0\le k\le 3} k^2\alpha_k = 0\times0+1\times(-1)+4\times0+9\times1 = 8 = 2\times4 = 2\left(0 \times \frac{1}{3} + 1 \times \frac{-2}{3} + 2 \times \frac{7}{3} + 3 \times 0\right) = 2 \sum_{0\le k\le 3} k\beta_k$*. The scheme is at least of order 2.*
- $i = 3$: $\sum_{0\le k\le 3} k^3\alpha_k = 0\times0+1\times(-1)+8\times0+27\times1 = 26 = 26\times\frac{26}{3} = 3\left(0 \times \frac{1}{3} + 1 \times \frac{-2}{3} + 4 \times \frac{7}{3} + 9 \times 0\right) = 3 \sum_{0\le k\le 3} k^2\beta_k$*. The scheme is at least of order 3.*
- $i = 4$: $\sum_{0\le k\le 3} k^4\alpha_k = 0 \times 0 + 1 \times (-1) + 16 \times 0 + 91 \times 1 = 80 \neq 72 = 4 \times 18$ $= 4\left(0 \times \frac{1}{3} + 1 \times \frac{-2}{3} + 8 \times \frac{7}{3} + 27 \times 0\right) = 4 \sum_{0\le k\le 3} k^3\beta_k$*. **The scheme is exactly of order 3**.*

vi. *As Scheme* (4) *is a 3-step scheme, we need **three initializing values** (to compute* z_4*, we need to know* z_3*,* z_2 *and* z_1*).*
As z_1 *must be an approximation of* $\hat{y}(t^1) = \hat{y}(0) = \frac{1}{2}$ *according to* (3)*, we choose* $\boxed{z_1 = \frac{1}{2}}$*. We now tackle the computation of* $z_2 \approx \hat{y}(t^2) = \hat{y}(\Delta t)$ *which we do not know. As Scheme* (4) *is of order 3, we must choose a value which is accurate at order 3, that is to say* $z_2 = \hat{y}(\Delta t) + \mathcal{O}(\Delta t^3)$*. To do so, we perform a Taylor expansion (Prop. 1.17):*

$$\hat{y}(\Delta t) = \hat{y}(0) + \Delta t\,\hat{y}'(0) + \frac{\Delta t^2}{2}\hat{y}''(0) + \mathcal{O}(\Delta t^3).$$

*$\hat{y}'(0)$ is deduce from ODE (**??**) and from the initial condition: $\hat{y}'(0) = f(\hat{y}(0)) = f\left(\frac{1}{2}\right)$. Likewise, to compute $\hat{y}''(0)$, we differentiate ODE (**??**) once to obtain*

$$\hat{y}''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[f\left(\hat{y}(t)\right)\right] = \hat{y}'(t)\frac{\mathrm{d}f}{\mathrm{d}y}\left(\hat{y}(t)\right) = \hat{y}'(t)f'\left(\hat{y}(t)\right) = f\left(\hat{y}(t)\right)f'\left(\hat{y}(t)\right)$$

Hence we set $\boxed{z_2 = \frac{1}{2} + \Delta t f\left(\frac{1}{2}\right) + \left(f\left(\frac{1}{2}\right) \times f'\left(\frac{1}{2}\right)\right)\frac{\Delta t^2}{2}}$. *We use the same method for z_3, we obtain* $\boxed{z_3 = \frac{1}{2} + (2\Delta t)f\left(\frac{1}{2}\right) + \left(f\left(\frac{1}{2}\right) \times f'\left(\frac{1}{2}\right)\right)\frac{(2\Delta t)^2}{2}}$.

vii. *The algorithm is the following:*

Algorithm 2 Resolution of ODE (3) with Scheme (4)

Data: N
$\Delta x = \frac{3}{N-1}$
z=[];
z(1)=1/2;
z(2)=expression given by the previous question;
z(3)=expression given by the previous question;
for i from 1 to N-3
$z(i+3) = z(i+1) + \Delta t\left(\frac{7}{3}f(z(i+2) - \frac{2}{3}f(z(i+1)) + \frac{1}{3}f(z(i))\right)$;
end

viii. *The Euler explicit scheme is a first order scheme and Scheme (4) is a third order scheme. Both schemes are explicit then from a computational point of view there are comparable. Thus **Schemes** (4) **is preferable** because it is more accurate.*

ix. *If $f(y) = 1$, the relation (4) becomes*

$$z_{n+3} - z_{n+1} = \Delta t\left(\frac{7}{3} \times 1 - \frac{2}{3} \times 1 + \frac{1}{3} \times 1\right) = 2\Delta t.$$

Since $f(y) = 1$, we have $f'(y) = 0$ and then, the Question 3(d)vi implies that

$$z_1 = \frac{1}{2}, \quad z_2 = \frac{1}{2} + \Delta t + \frac{\Delta t^2}{2}.$$

We have

$$\begin{cases} z_{n+2} = z_n + 2\Delta t \\ z_1 = \frac{1}{2} \end{cases}$$

then $\boxed{\forall n \geq 1,\ z_{2n+1} = \frac{1}{2} + n \times 2\Delta t}$. *We have*

$$\begin{cases} z_{n+2} = z_n + 2\Delta t \\ z_2 = \frac{1}{2} + \Delta t + \frac{\Delta t^2}{2} \end{cases}$$

then $\boxed{\forall n \geq 1,\ z_{2n} = \frac{1}{2} + \Delta t + \frac{\Delta t^2}{2} + (n-1) \times 2\Delta t}$.

Exercise 3 *We consider the PDE : Find ψ such that*

$$\begin{cases} \frac{\partial \psi}{\partial t}(t,x) - \frac{\sigma^2}{2}\frac{\partial^2 \psi}{\partial x^2}(t,x) = 0 & \forall (t,x) \in [0,T]\times[0,10] & \textit{(PDE)} \\ \psi(t,0) = 0 & \forall t \in [0,T] & \textit{(boundary condition in } x=0) \\ \psi(t,10) = 1 & \forall t \in [0,T] & \textit{(boundary condition in } x=10) \\ \psi(0,x) = \frac{x^2}{100} & \forall x \in [0,10] & \textit{(initial condition in } t=0). \end{cases} \tag{5}$$

1. *We have* $\boxed{t_1 = 0, \quad t_{N_t} = T, \quad x_1 = 0, \quad x_{N_x} = 10.}$
2. *The explicit Euler scheme writes*
$$\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} - \frac{\sigma^2}{2}\left(\frac{\psi_{j+1}^n - 2\psi_j^n + \psi_{j-1}^n}{(\Delta x)^2}\right) = 0.$$
3. *Adding discrete analogue of initial and boundary conditions (5) leads to the scheme*
$$\begin{cases} \psi_j^{n+1} = \psi_j^n + \Delta t\frac{\sigma^2}{2}\left(\frac{\psi_{j+1}^n - 2\psi_j^n + \psi_{j-1}^n}{(\Delta x)^2}\right) & 2 \le j \le N_x - 1, \quad 1 \le n \le N_t - 1 & \textit{(Euler explicit scheme)} \\ \psi_1^n = 0 & 2 \le n \le N_t & \textit{(boundary condition in } x=0) \\ \psi_{N_x}^n = 1 & 2 \le n \le N_t & \textit{(boundary condition in } x=10) \\ \psi_j^1 = \frac{x_j^2}{100} & 1 \le j \le N_x & \textit{(initial condition in } t=0) \end{cases} \tag{6}$$
Remark : for the sake of stability N_x and N_t must be chosen such that $\Delta\theta \le \frac{(\Delta x)^2}{\sigma^2}$.
4. *The implicit Euler scheme writes for* $2 \le j \le N_x - 1, \quad 1 \le n \le N_t - 1$ *:*
$$\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} - \frac{\sigma^2}{2}\left(\frac{\psi_{j+1}^{n+1} - 2\psi_j^{n+1} + \psi_{j-1}^{n+1}}{(\Delta x)^2}\right) = 0.$$
or equivalently
$$\psi_j^{n+1}\left(1 + \frac{\Delta t\,\sigma^2}{(\Delta x)^2}\right) - \frac{\Delta t\,\sigma^2}{2(\Delta x)^2}\psi_{j+1}^{n+1} - \frac{\Delta t\,\sigma^2}{2(\Delta x)^2}\psi_{j-1}^{n+1} = \psi_j^n \qquad 2 \le j \le N_x - 1$$
Initial and boundary conditions are the same as in (6). We have
$$\forall j \in [\![1, N_x]\!], \quad \psi_j^1 = \frac{x_j^2}{100}.$$
and
$$\psi_1^{n+1} = 0, \quad \psi_{N_x}^{n+1} = 1.$$
Then, to use this scheme one need to solve a linear system of the form
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{\Delta t\,\sigma^2}{(\Delta x)^2} & 1 + \frac{\Delta t\,\sigma^2}{(\Delta x)^2} & -\frac{\Delta t\,\sigma^2}{(\Delta x)^2} & 0 & \cdots & \cdots & 0 \\ 0 & -\frac{\Delta t\,\sigma^2}{(\Delta x)^2} & 1 + \frac{\Delta t\,\sigma^2}{(\Delta x)^2} & -\frac{\Delta t\,\sigma^2}{(\Delta x)^2} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -\frac{\Delta t\,\sigma^2}{(\Delta x)^2} & 1 + \frac{\Delta t\,\sigma^2}{(\Delta x)^2} & -\frac{\Delta t\,\sigma^2}{(\Delta x)^2} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_1^{n+1} \\ \psi_2^{n+1} \\ \vdots \\ \psi_j^{n+1} \\ \vdots \\ \psi_{N_x-1}^{n+1} \\ \psi_{N_x}^{n+1} \end{bmatrix} = \begin{bmatrix} \psi_1^n \\ \psi_2^n \\ \vdots \\ \psi_j^n \\ \vdots \\ \psi_{N_x-1}^n \\ \psi_{N_x}^n \end{bmatrix}.$$
The first line and the last line correspond to the boundary condition while other line correspond to the Euler explicit scheme.