Exam 2014/2015

No document is authorized except the lecture notes with no inscription inside. Cell phones and computers are forbidden.

The final exam has two pages and consists of two independent exercises. French students can answer the questions in french. Answers must rely on citations of numbered or named results from the lecture notes.

Exercise 1 *We consider the ordinary differential equation*

$$(t^2+1)\hat{y}'(t) = t\hat{y}(t)^2 - t. \tag{1}$$

1. *Determine two constant solutions of* (1).
2. *Assume that* $\hat{y}(0) = 0$. *Prove the existence of a unique maximal solution to* (1) *defined on* $J \subset \mathbb{R}$ *such that* $0 \in J$.
3. *Using Proposition 2.3 of the lecture notes, prove that the solution* $\hat{y}$ *of the previous question is defined on* $\mathbb{R}$ *(*$J = \mathbb{R}$*).*
4. *Set* $\hat{z} = \hat{y} - 1$. *Show that* $\hat{z}$ *satisfies*
$$(t^2+1)\hat{z}'(t) = t\big(2\hat{z}(t) + \hat{z}^2(t)\big). \tag{2}$$
5. *Set* $\hat{w} = \frac{1}{\hat{z}}$. *Show that* $\hat{w}$ *satisfies*
$$\hat{w}'(t) = -\frac{2t}{t^2+1}\hat{w}(t) - \frac{t}{t^2+1}. \tag{3}$$
6. *Solve* (3) *with the initial condition* $\hat{w}(0) = -1$.
7. *Deduce the solution of* (1) *with the initial condition* $\hat{y}(0) = 0$.
8. *Apply the explicit (forward) Euler scheme to* (1).
9. *Apply the implicit (backward) Euler scheme to* (1).

Exercise 2 *We aim at solving the autonomous (*i.e. f *does not depend on* t*) ordinary differential equation*

$$\begin{cases} \hat{y}'(t) = f\big(\hat{y}(t)\big), \\ \hat{y}(0) = \frac{1}{2}. \end{cases} \tag{4}$$

We assume that f *is of class* $\mathscr{C}^2(\mathbb{R})$.

1. *Justify that ODE* (4) *has a unique solution denoted* $\hat{y}$. *What is the regularity of* $\hat{y}$*?*
2. *Give the expression of the solution in the following cases:*
   (a) $f(y) = 1$;
   (b) $f(y) = 10y$.
3. *In the general case, we cannot provide an explicit expression. That is why we aim at constructing approximate values of the solution at some points. More precisely, we fixe some integer* $N \geq 2$ *and we set*
$$\forall n \in [\![1, N]\!],\ t^n = (n-1)\Delta t \quad \text{where} \quad \Delta t = \frac{3}{N-1}.$$
A numerical scheme is a method whose purpose is to compute y_n *which is an approximation of* $\hat{y}(t^n)$.

(a) Do we have $y_n = \hat{y}(t^n)$?

(b) What is t^1 equal to? t^2? t^N?

(c) i. Apply the explicit Euler scheme to ODE (4). Express y_{n+1} as a function of y_n.

ii. How many values do we need in order to initialize the sequence (y_n)?

iii. Write out the algorithm leading to the computation of the sequence (y_n).

iv. Recall the order of this scheme.

(d) We are interesting in the multi-step scheme

$$z_{n+3} - z_{n+1} = \Delta t \left(\frac{7}{3} f(z_{n+2}) - \frac{2}{3} f(z_{n+1}) + \frac{1}{3} f(z_n) \right). \tag{5}$$

i. Prove the consistency of Scheme (5).

ii. Study its stability.

iii. Deduce that this scheme is convergent.

iv. Is this scheme explicit or implicit? Justify your answer.

v. Determine the order of Scheme (5).

vi. How many values do we need in order to initialize the sequence (z_n)? Explain how to compute these initiliazing values.

vii. Write out the algorithm leading to the computation of the sequence (z_n).

viii. Which scheme would you recommend: Euler (Q. 3.(b)) or Scheme (5)?

ix. Apply Scheme (5) when $f(x) = 1$. Compute the exact expression of z_n for all n.

Exercise 3 *The aim of this exercice is to apply the Finite Difference Method (FDM) to find approximate solutions to a Partial Differential Equations (PDE) problem.*
We consider the PDE : Find ψ such that

$$\begin{cases} \frac{\partial \psi}{\partial t}(t,x) - \frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2}(t,x) = 0 & \forall (t,x) \in [0,T] \times [0,10] \quad \text{(PDE)} \\ \psi(t,0) = 0 & \forall t \in [0,T] \quad \text{(boundary condition in } x=0) \\ \psi(t,10) = 1 & \forall t \in [0,T] \quad \text{(boundary condition in } x=10) \\ \psi(0,x) = \frac{x^2}{100} & \forall x \in [0,10] \quad \text{(initial condition in } t=0), \end{cases} \tag{6}$$

where ψ is a function of two variables t and x.

We consider the discretization

$$\begin{cases} x_j = (j-1)\Delta x, \quad 1 \le j \le N_x \quad \Delta x = \frac{10}{N_x - 1} \\ t_n = (n-1)\Delta t \quad 1 \le n \le N_t \quad \Delta t = \frac{T}{N_t - 1} \end{cases}$$

for some $N_x > 1$ and $N_t > 1$.

1. *What are values of $t_1, t_{N_t}, x_1, x_{N_x}$?*

2. *Apply the Euler explicit scheme to PDE (6).*

3. *How do you consider the boundary conditions in your scheme?*

4. *Apply the Euler implicit scheme to PDE (6). To use this scheme one need to solve a linear system of the form*

$$A \left(\psi_j^{n+1} \right)_{1 \le j \le N_x} = \left(b_j^n \right)_{1 \le j \le N_x}$$

   Write matrix $A \in \mathscr{M}_{N_x}(\mathbb{R})$ and vector $b^n \in \mathbb{R}^{N_x}$ using the implicit Euler scheme and boundary conditions of (6).