Practical Work #5

1 Resolution of the Black & Scholes model in primitive variables

The Finite Difference Method (FDM) has been presented in the course. We aim at applying this method to the well-known Black & Scholes equation with constant volatility σ and constant interest rate r for the modelling of a European vanilla put option:

$$\begin{cases} \frac{\partial \tilde{P}}{\partial t}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 \tilde{P}}{\partial S^2}(t,S) + rS\frac{\partial \tilde{P}}{\partial S}(t,S) - r\tilde{P}(t,S) = 0, \\ \tilde{P}(T,S) = \max(0, K-S), \end{cases}$$

or equivalently (by means of the change of variables $P(t,S) = \tilde{P}(T-t,S)$)

$$\begin{cases} \frac{\partial P}{\partial t}(t,S) - \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}(t,S) - rS\frac{\partial P}{\partial S}(t,S) + rP(t,S) = 0, & \text{(1a)} \\ P(0,S) = \max(0, K-S). & \text{(1b)} \end{cases}$$

We set for the present study

$$K = 100, \quad T = 1, \quad \sigma = 0.2, \quad r = 0.04, \quad b = \frac{1}{2} - \frac{r}{\sigma^2}, \quad a = -r - \frac{\sigma^2 b^2}{2}, \quad \underline{S} = 0.1, \quad \overline{S} = 1000$$

and consider the discretization

$$\begin{cases} S_j = \underline{S} + (j-1)\Delta S, & 1 \le j \le N_S & \Delta S = \frac{\overline{S}-\underline{S}}{N_S-1} \\ t_n = (n-1)\Delta t & 1 \le n \le N_t & \Delta t = \frac{T}{N_t-1} \end{cases}$$

for some $N_S > 1$ and $N_t > 1$.

1. What are values of $t_1, t_{N_t}, S_1, S_{N_S}$?

2. We propose the following explicit numerical scheme based on the explicit Euler scheme

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} - \frac{\sigma^2 (S_j)^2}{2}\left(\frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{(\Delta S)^2}\right) + rS_j\frac{P_{j+1}^n - P_{j-1}^n}{2\Delta S} - rP_j^n = 0,$$

   adding discrete analogue of initial and boundary conditions leads to the scheme

$$\begin{cases} P_j^{n+1} = P_j^n + \frac{\sigma^2 \Delta t (S_j)^2}{2}\left(\frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{(\Delta S)^2}\right) - r\Delta t S_j \frac{P_{j+1}^n - P_{j-1}^n}{2\Delta S} + r\Delta t P_j^n, & 2 \le j \le N_x - 1 \quad 1 \le n \le N_t - 1 \text{(scheme)} \\ P_1^n = Ke^{-rt_n} & 2 \le n \le N_t \quad \text{(boundary condition in } S = \underline{S}) \\ P_{N_S}^n = 0 & 2 \le n \le N_t \quad \text{(boundary condition in } S = \overline{S}) \\ P_j^1 = \max(0, K - S_j) & 1 \le j \le N_S \quad \text{(initial condition in } t = 0) \end{cases}$$

   Implement this scheme to compute P_j^n for all $1 \le j \le N_x$ and $1 \le n \le N_t$.

3. For the sake of stability the value of Δt must be chosen small enough for a given ΔS. Find out by means of numerical simulations a suitable value for N_t when $N_S = 100$, $N_S = 1000$ and $N_S = 10000$ (increase the value of N_t until you obtain stability).

4. Plot the solution at time T (ie plot $P_j^{N_t}$ as a function of S_j) and comment numerical results.