Practical Work #3

The aim of this session is to apply numerical methods to find approximate solutions to different Ordinary Differential Equations problems and to cope with classical numerical issues.

1 A linear equation

We consider the ODE

$$\begin{cases} \hat{y}'(t) = \hat{y}(t), \\ \hat{y}(0) = 1. \end{cases} \tag{1}$$

The exact solution is given by $\hat{y}(t) = e^t$. We set $t^n = (n-1)\Delta t$, $1 \le n \le N$ and $\Delta t = \frac{1}{N-1}$.

1. What are the initial time t^1 and final time t^N associated to the discretization? Plot the image of the vector $(t^n)_{1\le n\le N}$ by the function $\hat{y}$.
2. Implement the explicit Euler scheme and the enhanced Euler scheme. We use those schemes to compute $(y_{num}(t^n))_{1\le n\le N}$ that are approximations of $(\hat{y}(t^n))_{1\le n\le N}$.
3. Show the curves $t \mapsto \big(t, y_{num}(t)\big)$ for $N = 100$, $N = 200$, $N = 500$ and $N = 1000$ as well as $t \mapsto \big(t, \hat{y}(t)\big)$. What do you observe ?

2 A nonlinear equation

We consider the ODE

$$\begin{cases} t\hat{y}(t)\hat{y}'(t) = \hat{y}^2(t) - 1, \\ \hat{y}(1) = 2. \end{cases} \tag{2}$$

The exact solution $\hat{y}$ is given by $\hat{y}(t) = \sqrt{1+3t^2}$. Set $t^n = 1 + (n-1)\Delta t$, $1 \le n \le N$ and $\Delta t = \frac{2}{N-1}$.

1. What are the initial time t^1 and final time t^N associated to the discretization? Plot the image of the vector $(t^n)_{1\le n\le N}$ by the function $\hat{y}$.
2. Implement the explicit Euler scheme and the enhanced Euler scheme. We use those schemes to compute $(y_{num}(t^n))_{1\le n\le N}$ that are approximations of $(\hat{y}(t^n))_{1\le n\le N}$.
3. Show the curves $t \mapsto \big(t, y_{num}(t)\big)$ for $N = 100$, $N = 200$, $N = 500$ and $N = 1000$ as well as $t \mapsto \big(t, \hat{y}(t)\big)$.
4. For these four values of N, compute $e_{num}(N) = \max_{1\le n\le N} |y_n - \hat{y}(t^n)|$. Plot $e_{num}(N)$ as a function of N for the two schemes and discuss their performance. Then plot $ln(e_{num}(N))$ as a function of $ln(N)$ and determine the order of the explicit Euler scheme and the enhanced Euler scheme.

3 Preys vs predators

In the Lotka–Volterra model

$$\begin{cases} x'(t) = x(t)\big(3 - y(t)\big), & x(0) = 1, \\ y'(t) = y(t)\big(x(t) - 2\big), & y(0) = 2. \end{cases} \tag{3}$$

x and y denote respectively the rate of preys and predators in a closed area. Even if the exact solution is not explicitly known, it is proven that x and y are periodic functions of time. Moreover, the Hamiltonian

$$H(x, y) = x - 2\ln x + y - 3\ln y$$

is preserved (ie $H(x(t), y(t)) = H(x(0), y(0))$ for all $t \geq 0$).
Set $t^n = (n-1)\Delta t$, $1 \leq n \leq N$ and $\Delta t = \frac{10}{N-1}$.

1. Implement the explicit Euler scheme and the Heun scheme.
2. Draw $t \mapsto x(t)$ and $t \mapsto y(t)$. Is the result as expected?
3. Plot the graphs $t \mapsto H\big(x(t), y(t)\big)$ for both schemes as well as the theoretical value.
4. We propose the following scheme (symplectic Euler scheme)

$$\begin{cases} \dfrac{x_{n+1} - x_n}{\Delta t} = x_{n+1}(3 - y_n), \\ \dfrac{y_{n+1} - y_n}{\Delta t} = y_n(x_{n+1} - 2). \end{cases}$$

   Does the conclusion of Questions 2 and 3 still holds?