Quantitative Risk Analysis
Master of Financial Engineering - M2
Intermediate Exam - Duration : 1 hour
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No documents - <u>You can use a calculator</u>

> Please report clear and detailed answers explicitly, as well as all the formulas you use and all the theorems you invoke.

Problem 1 :

1) Give the mathematical definition of VaR_α and explain its meaning.
Solution : We have

$$VaR_\alpha(X) = \min\{g : \ P(X \leq g) \geq \alpha\}.$$

This is the maximal loss at confidence level α. In other words, there is a probability less than $1-\alpha$ that the loss be larger than $VaR_\alpha(X)$.

2) Give the mathematical definition of CVaR_α and explain its meaning.
Solution : In the continuous case, we have

$$CVaR_\alpha(X) = E\big(X|X \geq VaR_\alpha\big) = \frac{E\Big(X\mathbf{1}_{[VaR_\alpha(X),+\infty[}(X)\Big)}{P\big(X \geq VaR_\alpha(X)\big)}$$
$$= \frac{1}{1-\alpha}\int_{VaR_\alpha(X)}^{+\infty} zf(z)dz$$

because $P\big(X \geq VaR_\alpha(X)\big) = 1-\alpha$. Then, $CVaR_\alpha(X)$ is the average loss above $VaR_\alpha(X)$. By remarking that

$$z = VaR_\gamma(X) \Leftrightarrow \gamma = P(X \leq z),$$

we can change the variable z into γ $\left(\frac{d\gamma}{dz} = \frac{d}{dz}P(X \leq z) = f(z)\right)$ and write $CVaR_\alpha(x)$ as

$$CVaR_\alpha(X) = \frac{1}{1-\alpha}\int_{VaR_\alpha(X)}^{+\infty} zf(z)dz = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\gamma(X)d\gamma.$$

The $CVaR_\alpha$ can also be seen as the average of the VaR_γ for $\gamma \in [\alpha, 1]$.

Problem 2 : We consider an investment for which the price at time $t=0$ is 100 euros and the value at time $T>0$ is a random variable V. The density of the random variable V is given by :

$$\begin{cases} P(V=80) = 0.01, \\ P(V=90) = 0.03, \\ P(V=95) = 0.26, \\ P(V=100) = 0.30, \\ P(V=105) = 0.10, \\ P(V=106) = 0.30. \end{cases}$$

1) Write the loss X at time T as a function of V.
Solution : We have

$$X = 100 - V.$$

2) Give the density of the random variable X.
Solution : The density of the random variable X is given by :

$$\begin{cases} P(X = 100 - 80 = 20) = 0.01, \\ P(X = 100 - 90 = 10) = 0.03, \\ P(X = 100 - 95 = 5) = 0.26, \\ P(X = 100 - 100 = 0) = 0.30, \\ P(X = 100 - 105 = -5) = 0.10, \\ P(X = 100 - 106 = -6) = 0.30. \end{cases}$$

3) Compute $\mathrm{VaR}_{0.9}(X)$. Explain your method.
Solution : We have

$$P(X \leq 20) = 1 \geq 0.95.$$

We also have

$$P(X \leq 10) = 1 - P(X = 20) = 1 - 0.01 = 0.99 \geq 0.95$$

and

$$P(X \leq 5) = 1 - 0.01 - 0.03 = 0.96 \geq 0.95$$

and

$$P(X \leq 0) = 1 - 0.01 - 0.03 - 0.26 = 0.96 - 0.26 = 0.7 < 0.95$$

then

$$VaR_{0.95}(X) = 5.$$

4) What is the biggest value α such that $\mathrm{VaR}_\alpha(X) = 0$?
Solution : We have

$$P(X \leq 0) = 0.3 + 0.3 + 0.1 = 0.7$$

then the biggest α such that $\mathrm{VaR}_\alpha(X) = 0$ is 0.7.

Problem 3 : Consider two independent securities Y_1 and Y_2. At time $t = 0$, these two securities cost 20 euros. At time $t = T$, the value of security $i = 2$ is exponentially distributed with an average of 20 euros. At time $t = T$ the value of security $i = 1$ is 22 with a probability of 90% and 15 with a probability of 10%. We recall that the density of the exponential distribution with average μ is

$$f(x) = \begin{cases} \frac{1}{\mu} e^{-\frac{x}{\mu}}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

1) Give the density of the first security $i = 1$.
Solution : We have

$$P(Y_1 = 22) = 0.9 \quad \text{and} \quad P(Y_1 = 15) = 0.1.$$

2) Compute $VaR_{0.95}$ for the loss at time $t = T$ of security $i = 1$.
Solution : The loss X_1 at time $t = T$ of the security $i = 1$ is given by

$$X_1 = 20 - Y_1$$

and its density is

$$P(X_1 = 20 - 22 = -2) = 0.9 \quad \text{and} \quad P(X_1 = 20 - 15 = 5) = 0.1.$$

We have

$$P(X_1 \leq -2) = 0.9 < 0.95 \quad \text{and} \quad P(X_1 \leq 5) = 1 \geq 0.95$$

and then

$$VaR_{0.95}(X_1) = 5.$$

3) Compute the cumulative function of an exponential random variable with average μ.
Solution : The cumulative F_2 is given for all $x \in \mathbb{R}$ by

$$\begin{aligned} F_2(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \begin{cases} \int_{-\infty}^{x} 0dt = 0, & \text{if } x \leq 0, \\ \int_0^x \frac{1}{\mu} e^{-\frac{t}{\mu}} dt = \left[-e^{-\frac{t}{\mu}} \right]_0^x = 1 - e^{-\frac{x}{\mu}}, & \text{otherwise.} \end{cases} \end{aligned}$$

4) Compute $VaR_{0.95}$ for the loss at time $t = T$ of security $i = 2$.
Solution : The loss at a time $t = T$ os security $i = 2$ is

$$X_2 = 20 - Y_2$$

where Y_2 is an exponential law. We look for $g \in \mathbb{R}$ such that

$$\begin{aligned} P(X_2 \leq g) \geq 0.95 &\Leftrightarrow P(20 - Y_2 \leq g) \geq 0.95 \\ &\Leftrightarrow P(Y_2 \geq 20 - g) \geq 0.95 \Leftrightarrow 1 - P(Y_2 \leq 20 - g) \geq 0.95 \\ &\Leftrightarrow 1 - F_2(20 - g) \geq 0.95 \Leftrightarrow 1 - \left(1 - e^{-\frac{20-g}{\mu}}\right) \geq 0.95 \\ &\Leftrightarrow e^{-\frac{20-g}{\mu}} \geq 0.95 \Leftrightarrow -\frac{20 - g}{\mu} \geq \ln(0.95) \Leftrightarrow \frac{g - 20}{\mu} \geq \ln(0.95) \\ &\Leftrightarrow g \geq \mu \ln(0.95) + 20 \approx 18.97. \end{aligned}$$

Then,

$$VaR_{0.95}(X_2) = 18.97.$$

5) Compute the $0.95 - VaR$ at time $t = T$ of a portfolio made up of these two securities.
Solution : We look for $g \in \mathbb{R}$ such that

$$P(X_1 + X_2 \leq g) \geq 0.95.$$

Since, X_1 takes only the values -2 or 5, we have for all $g \in \mathbb{R}$,

$$\begin{aligned} P(X_1 + X_2 \leq g) &= P\Big([\{X_1 + X_2 \leq g\} \cap \{X_1 = -2\}] \\ &\qquad \cup [\{X_1 + X_2 \leq g\} \cap \{X_1 = 5\}] \Big) \\ &= P\left(\{X_1 + X_2 \leq g\} \cap \{X_1 = -2\}\right) \\ &\qquad + P\left(\{X_1 + X_2 \leq g\} \cap \{X_1 = 5\}\right) \end{aligned}$$

because the sets are disjointed. Moreover, we have for all $g \in \mathbb{R}$,

$$\begin{aligned} P(X_1 + X_2 \leq g) &= P\left(\{X_1 + X_2 \leq g\} \cap \{X_1 = -2\}\right) + P\left(\{X_1 + X_2 \leq g\} \cap \{X_1 = 5\}\right) \\ &= P\left(\{X_2 \leq g + 2\} \cap \{X_1 = -2\}\right) + P\left(\{X_2 \leq g - 5\} \cap \{X_1 = 5\}\right) \\ &= P\left(X_2 \leq g + 2\right) \times P\left(X_1 = -2\right) + P\left(X_2 \leq g - 5\right) \times P\left(X_1 = 5\right) \end{aligned}$$

because X_1 and X_2 are independent. Then, we obtain for all $g \in \mathbb{R}$,

$$\begin{aligned} P(X_1 + X_2 \leq g) &= 0.9 \times P(X_2 \leq g+2) + 0.1 \times P(X_2 \leq g-5) \\ &= 0.9 \times P(20 - Y_2 \leq g+2) + 0.1 \times P(20 - Y_2 \leq g-5) \\ &= 0.9 \times P(Y_2 \geq 18 - g) + 0.1 \times P(Y_2 \geq 25 - g) \\ &= 0.9 \times (1 - F_2(18-g)) + 0.1 \times (1 - F_2(25-g)). \end{aligned}$$

We look for $g \in \mathbb{R}$ such that

$$\begin{aligned} P(X_1 + X_2 \leq g) \geq 0.95 &\Leftrightarrow 1 - 0.9 \times F_2(18-g) - 0.1 \times F_2(25-g) \geq 0.95 \\ &\Leftrightarrow 0.9 \times F_2(18-g) + 0.1 \times F_2(25-g) \leq 0.05 \end{aligned}$$

That, it seams natural to firstly try to found $g \leq 18$ because

$$g \mapsto 0.9 \times F_2(18-g) + 0.1 \times F_2(25-g)$$

is an decreasing function from $\mathbb{R}$ to $[0,1]$. If $g \leq 18$, we have

$$\begin{aligned} P(X_1 + X_2 \leq g) \geq 0.95 &\Leftrightarrow 0.9 \times F_2(18-g) + 0.1 \times F_2(25-g) \leq 0.05 \\ &\Leftrightarrow 0.9\left(1 - e^{-\frac{18-g}{\mu}}\right) + 0.1\left(1 - e^{-\frac{25-g}{\mu}}\right) \leq 0.05 \\ &\Leftrightarrow 1 - 0.9e^{-\frac{18-g}{\mu}} - 0.1e^{-\frac{25-g}{\mu}} \leq 0.05 \\ &\Leftrightarrow 0.9e^{-\frac{18-g}{\mu}} + 0.1e^{-\frac{25-g}{\mu}} \geq 0.95 \\ &\Leftrightarrow \left(0.9e^{-\frac{18}{\mu}} + 0.1e^{-\frac{25}{\mu}}\right) e^{\frac{g}{\mu}} \geq 0.95 \Leftrightarrow e^{\frac{g}{\mu}} \geq \frac{0.95}{0.9e^{-\frac{18}{\mu}} + 0.1e^{-\frac{25}{\mu}}} \\ &\Leftrightarrow \frac{g}{\mu} \geq \ln\left(\frac{0.95}{0.9e^{-\frac{18}{\mu}} + 0.1e^{-\frac{25}{\mu}}}\right) \\ &\Leftrightarrow g \geq \mu \ln\left(\frac{0.95}{0.9e^{-\frac{18}{\mu}} + 0.1e^{-\frac{25}{\mu}}}\right) \approx 17.57 (< 18). \end{aligned}$$

Then,

$$VaR_{0.95}(X_1 + X_2) = 17.57.$$

6) Give the definition of a sub-additive risk measure ? Can VaR be a sub-additive variable ?
Solution : Let X and Y be the loss at time T of two portfolios. We say that the risk measure ρ satisfies the sub-additivity property if

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

In our case, we have

$$VaR_{0.95}(X_1 + X_2) = 17.57 < 23.95 = 5 + 18.95 = VaR_{0.95}(X_1) + VaR_{0.95}(X_2).$$

then the VaR could be sub-addivitive. However, you saw in some examples of the course that VaR is not a sub-additive risk measure.