**Quantitative Risk Analysis**
*Master of Financial Engineering - M2*
Final exam - Duration : 2 hours
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## No documents - <u>You can use a calculator</u>

> Please report clear and detailed answers explicitly, as well as all the formulas you use and all the theorems you invoke.

**Problem 1 :**
Consider a security whose **prices** $y_0$, ..., $y_{10}$ during the last 11 days have been :

$$y_0 = 10, \quad y_1 = 8, \quad y_2 = 5, \quad y_3 = 12, \quad y_4 = 4, \quad y_5 = 2,$$
$$y_6 = 1, \quad y_7 = 5, \quad y_8 = 1, \quad y_9 = 3, \quad y_{10} = 2.$$

The daily loss is the loss on the price of the security between a given day and the following day.

1) Compute the 10 daily looses $x_1, ..., x_{10}$ that correspond to the 11 prices of the security.
**Solution :** The 10 daily losses are given by the formula

$$\forall i \in [\![1, 10]\!], \quad x_i = y_{i-1} - y_i.$$

Then, we obtain for the last 10 days

$$x_1 = 10 - 8 = 2, \quad x_2 = 8 - 5 = 3, \quad x_3 = 5 - 12 = -7, \quad x_4 = 12 - 4 = 8,$$
$$x_5 = 4 - 2 = 2, \quad x_6 = 2 - 1 = 1, \quad x_7 = 1 - 5 = -4, \quad x_8 = 5 - 1 = 4,$$
$$x_9 = 1 - 3 = -2, \quad x_{10} = 3 - 2 = 1.$$

2) Compute the historic VaR of these yields with $\alpha = 0.90$.
**Solution :** We sort the sequence :

$$x_3 = -7, \quad x_7 = -4, \quad x_9 = -2, \quad x_6 = 1, \quad x_{10} = 1,$$
$$x_5 = 2, \quad x_1 = 2, \quad x_2 = 3, \quad x_8 = 4, \quad x_4 = 8.$$

We have

$$\frac{\#\{i : x_i \leq 4\}}{10} = \frac{9}{10} = 0.9 \geq 0.9$$

and

$$\frac{\#\{i : x_i \leq 3\}}{10} = \frac{8}{10} = 0.8 < 0.9$$

then the historic 0.9-$VaR$ is 4.

3) Compute the historic CVaR of these yields with $\alpha = 0.90$.
**Solution :** The historic 0.9-$CVaR$ is the average of the the yields bigger than the 0.9-$VaR$. We have

$$\frac{4 + 8}{2} = 6.$$

then the historic 0.9-$CVaR$ is 6.

**Problem 2 :**

Consider a security whose price at time $t = 0$ is 100 euros and whose **price** at time $T > 0$ is a random variable $X$ exponentially distributed with an average of 105 euros.
We recall that the density of the exponential distribution with average $\mu$ is

$$f(x) = \begin{cases} \frac{1}{\mu} e^{-\frac{x}{\mu}}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

1) Compute the 0.95-VaR of the loss at time $T$.
**Solution :** The loss $L$ at time $T$ is given by

$$L = 100 - X.$$

We look for $g \in \mathbb{R}$ such that

$$P(L \leq g) \geq \alpha = 0.95.$$

For all $g \in \mathbb{R}$, we have

$$\begin{aligned} P(L \leq g) = P(100 - X \leq g) = P(X \geq 100 - g) &= \int_{100-g}^{+\infty} f(x)dx \\ &= \begin{cases} \int_{100-g}^{+\infty} \frac{1}{\mu} e^{-\frac{x}{\mu}} dx = \left[ -e^{-\frac{x}{\mu}} \right]_{100-g}^{+\infty} = e^{-\frac{100-g}{\mu}}, & \text{if } 100 - g \geq 0 \\ \int_{0}^{+\infty} \frac{1}{\mu} e^{-\frac{x}{\mu}} dx = 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Then,

$$\begin{aligned} P(L \leq g) \geq \alpha \Leftrightarrow e^{-\frac{100-g}{\mu}} \geq \alpha \Leftrightarrow -\frac{100-g}{\mu} \geq \ln(\alpha) \\ \Leftrightarrow g - 100 \geq \mu \ln(\alpha) \Leftrightarrow g \geq \mu \ln(\alpha) + 100 \end{aligned}$$

and we obtain that for all $\alpha \in ]0, 1[$,

$$VaR_{\alpha}(L) = 105 \times \ln(\alpha) + 100.$$

Then,

$$VaR_{0.95}(L) = 105 \times \ln(0.95) + 100 = 94.61.$$

2) Compute the 0.95-CVaR of the loss at time $T$.
**Solution :** Since for all $\alpha \in ]0, 1[$,

$$VaR_{\alpha}(L) = 105 \times \ln(\alpha) + 100,$$

we obtain

$$\begin{aligned} CVaR_{0.95}(L) &= \frac{1}{1 - 0.95} \int_{0.95}^{1} VaR_{\alpha}(L) d\alpha = \frac{1}{0.05} \int_{0.95}^{1} 105 \times \ln(\alpha) + 100 \, d\alpha, \\ &= \frac{1}{0.05} \left[ 105 \times \alpha \ln(\alpha) - 105\alpha + 100\alpha \right]_{0.95}^{1} = \frac{1}{0.05} \left[ 105 \times \alpha \ln(\alpha) - 5\alpha \right]_{0.95}^{1} \\ &= \frac{-5 - 105 \times 0.95 \times \ln(0.95) + 5 \times 0.95}{0.05} \\ &= 97.33. \end{aligned}$$

**Problem 3** :
Assume we need to estimate

$$\theta = P(X \geq 3)$$

where $X$ follows a standard normal law.
We recall that the density of the normal distribution with average $\mu$ and standard deviation is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right)$$

We will use a Monte-Carlo method to estimate $\theta$.

1) Write $\theta$ as

$$\theta = E_f(h(X))$$

for some function $h : \mathbb{R} \to \mathbb{R}$.
**Solution :** We have

$$\begin{aligned}\theta = P(X \geq 3) &= \int_3^{+\infty} f(x)dx \\ &= \int_{-\infty}^{+\infty} \mathbf{1}_{[3;+\infty[}(x) f(x)dx \\ &= E\left(\mathbf{1}_{[3;+\infty[}(X)\right),\end{aligned}$$

then, for all $x \in \mathbb{R}$,

$$h(x) = \mathbf{1}_{[3;+\infty[}(x) = \begin{cases} 1, & \text{if } x \geq 3, \\ 0, & \text{otherwise.}\end{cases}$$

2) Assume that

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1$$

are realizations following density $f$. Give an estimation $\theta_3$ of $\theta$ by Monte-Carlo using these three realizations.
**Solution :** The method can be split in two steps :
— We compute $h(x_1)$, $h(x_2)$ and $h(x_3)$. We have

$$h_1 = h(x_1) = 0, \quad h_2 = h(x_2) = 0, \quad h_3 = h(x_3) = 0.$$

— We do the average of value of $h_i$

$$\theta_3 = \frac{h_1 + h_2 + h_3}{3} = 0.$$

3) Give a confidence interval at level $p = 95\%$ for $\theta$. What is the problem ?
**Solution :** With the central limit theorem, we have

$$P\left(Z \leq \frac{\theta_n - \theta}{s/\sqrt{n}} \leq Z\right) = 0.95 \Leftrightarrow P\left(\theta_n - Z \times \frac{s}{\sqrt{n}} \leq \theta \leq \theta_n + Z \times \frac{s}{\sqrt{n}}\right) = 0.95.$$

Then, a 95% confidence interval for $\theta$ is given by

$$\theta \in \left[\theta_3 - 1.96 \times \frac{s_3}{\sqrt{3}}, \theta_3 + 1.96 \times \frac{s_3}{\sqrt{3}}\right]$$

where we need to compute $s_3$. We have

$$s_3 = \sqrt{\frac{(h_1 - \theta_3)^2 + (h_2 - \theta_3)^2 + (h_3 - \theta_3)^2}{3}} = 0,$$

then we obtain

$$\theta \in [0, 0].$$

4) Using the importance sampling method and the density $g(x)$ of a normal law with average

$\mu$ and standard deviation $\sigma = 1$, write $\theta$ as

$$\theta = E_g\left(\bar{h}(X)\right)$$

for some function $\bar{h} : \mathbb{R} \to \mathbb{R}$.

**Solution :** We have

$$\theta = E_f\left(\mathbf{1}_{[3;+\infty[}(X)\right) = \int_{-\infty}^{+\infty} \mathbf{1}_{[3;+\infty[}(x) f(x) dx$$
$$= \int_{-\infty}^{+\infty} \mathbf{1}_{[3;+\infty[}(x) \frac{f(x)}{g(x)} g(x) dx = E_g\left(\mathbf{1}_{[3;+\infty[}(X) \frac{f(X)}{g(X)}\right),$$

then for all $x \in \mathbb{R}$,

$$\bar{h}(x) = \mathbf{1}_{[3;+\infty[}(x) \frac{f(x)}{g(x)}.$$

This function is well defined because the support of $g$ is $\mathbb{R}$ and the support of $f$ is $\mathbb{R}$.

5) Describe the maximum principle method and choose $\mu$ such that the maximum principle is satisfied.

**Solution :** The maximum principle is a method to choose the density $g$. We choose $g$ so that

$$\begin{cases} x \mapsto h(x)f(x) \\ x \mapsto g(x) \end{cases} \quad \text{are maximal for the same value } x^*.$$

We have for all $x \in \mathbb{R}$,

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2}\right)$$

then $g$ is maximal if $x^* = \mu$. On the other hand, for all $x \in \mathbb{R}$,

$$h(x)g(x) = \mathbf{1}_{[3;+\infty[}(x) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

then $g$ is maximal for $x^* = 3$. One choose $\mu = x^* = 3$ to reduce the variance.

6) Assume that

$$x_1 = -2, \quad x_2 = 3, \quad x_3 = 4$$

are realizations following density $g$. Give an estimation $\theta_3$ of $\theta$ by Monte-Carlo using these three realizations.

**Solution :** The method can be split in two steps :

— We compute $\bar{h}(x_1)$, $\bar{h}(x_2)$ and $\bar{h}(x_3)$. We have

$$\bar{h}_1 = \bar{h}(x_1) = 0, \quad \bar{h}_2 = \bar{h}(x_2) = \frac{\exp\left(\frac{-x_2^2}{2}\right)}{\exp\left(\frac{-(x_2-3)^2}{2}\right)}, \quad \bar{h}_3 = \bar{h}(x_3) = \frac{\exp\left(\frac{-x_3^2}{2}\right)}{\exp\left(\frac{-(x_3-3)^2}{2}\right)}.$$

— We do the average of value of $\bar{h}_i$

$$\theta_3 = \frac{\bar{h}_1 + \bar{h}_2 + \bar{h}_3}{3} = 0.0039.$$

7) Give a confidence interval at level $p = 95\%$ for $\theta$ based on the importance sampling method.

**Solution :** The 95% confidence interval for $\theta$ is given by

$$\theta \in \left[\theta_3 - 1.96 \times \frac{s_3}{\sqrt{3}}, \theta_3 + 1.96 \times \frac{s_3}{\sqrt{3}}\right]$$

where we need to compute $s_3$. We have

$$s_3 = \sqrt{\frac{(\bar{h}_1 - \theta_3)^2 + (\bar{h}_2 - \theta_3)^2 + (\bar{h}_3 - \theta_3)^2}{3}} = 0.0063,$$

then we obtain

$$\theta \in [-0.0032, 0.0110] \, .$$