Discrete pricing models - Notions

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This introductory part is designed to recall the simplest pricing models, and to introduce the notations that will be used later. The emphasis is on discrete modeling since computers will handle such models more efficiently. In this text the symbols ^{T} denote the transpose of the vector or matrix that forgoes.

1 Notations and definitions for single period models

1.1 Notations

Notations : Consider a time period beginning at instant $t = 0$ (current time), and ending at instant $t = 1$. We will consider a market made up of $N + 1$ securities whose prices evolve from $t = 0$ to $t = 1$. As $t = 0$ all the prices are known to us (since $t = 0$ is the current time) and at $t = 1$, their prices are unknown but there are m possible states ω_1, ..., ω_m. It is unknown which state will occur, but the price of every security is known in each of the possible states. Call $S_0^{(i)}$ the value of security $i \in [\![0, N]\!]$ at instant $t = 0$, and $S_1^{(i)}(\omega_j)$ the value of security $i \in [\![0, N]\!]$ at $t = 1$ if state $j \in [\![1, m]\!]$ occurs. Further call $P = (p_1, ..., p_m)$ the probability distribution of the possible states ω_1, ..., ω_m at instant $t = 1$. It can be assumed without loss of generality that $p_j > 0$ for every j (otherwise state ω_j would never occur and can be removed from the model).

1.2 Arbitrage

Definition 1.1. ***(type A arbitrage)****: A type A arbitrage is an investment so that:*

1. *An immediate profit is obtained at time $t = 0$,*
2. *There is no cost due at time $t = 1$.*

Example: A type A arbitrage would be somebody walking up to you on the street, giving you a positive amount of cash, and asking for nothing in return, either then or in the future.

Definition 1.2. ***(type B arbitrage)****: A type B arbitrage is an investment so that:*

1. *The cost at time $t = 0$ is non-positive,*
2. *The probability to produce a profit at time $t = 1$ is non-zero (hence this probability is positive),*
3. *The probability to produce a loss at time $t = 1$ is zero.*

Example: A type B arbitrage would be a stock that costs nothing, but that will possibility generate dividend income in the future.

Fundamental assumption: when modeling financial markets, it is assumed that there never is an arbitrage situation. Indeed the forces of the market quickly make such situations disappear.

1.3 Linear pricing

Definition 1.3. ***(Linear pricing)****: If $S_0^{(0)} \in \mathbb{R}$ and $S_0^{(1)} \in \mathbb{R}$ are the values of two securities ($i = 0$ and $i = 1$) at instant $t = 0$, and if the values of these securities at instant t=1 are row vectors $S_1^{(0)} \in \mathbb{R}^m$ and $S_1^{(1)} \in \mathbb{R}^m$ (the j-th entry of these vectors is the value of securities $i = 0$ and $i = 1$ in state ω_j. Hence these vectors have m entries), one will say that the linear pricing hypothesis is satisfied if for every pair of real numbers $\alpha_0 \in \mathbb{R}$ and $\alpha_1 \in \mathbb{R}$,*

$$\alpha_0 S_0^{(0)} + \alpha_1 S_0^{(1)}$$

is the value at $t = 0$ of any security whose value at $t = 1$ is

$$\alpha_0 S_1^{(0)} + \alpha_1 S_1^{(1)}.$$

Proposition 1.4. ***(link between arbitrage and linear pricing)**: If there is no type A arbitrage, the linear pricing hypothesis is necessarily satisfied.*

Proof. exercise!! □

1.4 Elementary securities, attainability, portfolio, complet market and state prices

Definition 1.5. ***(Elementary security)**: an elementary security is a security whose value at time $t = 1$ is $e_j = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^m$. In other words, the value of this security at time $t = 1$ is 1 in state ω_j and 0 in every other state.*

Definition 1.6. ***(attainable security, equivalent portfolio)**: A security of value $X \in \mathbb{R}^m$ (recall that X is a row vector with m entries) at time $t = 1$ is attainable if it can be replicated by a portfolio $\theta = (\theta_0, ..., \theta_N) \in \mathbb{R}^{N+1}$ (this portfolio is modeled as a row vector, the quantity of security i it contained in this portfolio being θ_i) so that:*

$$X = \theta\, S_1 = \left(\sum_{i=0}^{N} \theta_i\, S_1^{(i)}(\omega_j) \right)_{1 \le j \le m}.$$

In this expression, S_1 is the $(N+1) \times m$ matrix whose entry in row i and column j is $S_1^{(i)}(\omega_j)$. Vector θ is also called a portfolio equivalent to a security of value X at instant $t = 1$.

Example: Assume that there are 4 possible states $\omega_1, ..., \omega_4$ at instant $t = 1$, and the market is made up of three securities whose values at time $t = 0$ are respectively 1.0194, 3.4045 and 2.4917. In this case, $m = 4$ and $N = 2$. Further assume that vectors $S_1(\omega_j)$ are as follows:

$$\begin{aligned} S_1(\omega_1) &= (1.03, 3, 2)^T, \\ S_1(\omega_2) &= (1.03, 4, 1)^T, \\ S_1(\omega_3) &= (1.03, 2, 4)^T, \\ S_1(\omega_4) &= (1.03, 5, 2)^T, \end{aligned}$$

then

$$S_0 = \begin{pmatrix} 1.0194 \\ 3.4045 \\ 2.4917 \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} 1.03 & 1.03 & 1.03 & 1.03 \\ 3 & 4 & 2 & 5 \\ 2 & 1 & 4 & 2 \end{pmatrix}.$$

For instance, at time $t = 1$ and in state ω_3, the values of the three securities are respectively 1.03, 2 and 4. A security of value $X = (7.47, 6.97, 9.97, 10.47)$ at instant $t = 1$ is attainable since $X = \theta\, S_1$ with $\theta = (-1, 1.5, 2)$. In other words, $\theta = (-1, 1.5, 2)$ is a portfolio equivalent to this security.

Definition 1.7. ***(complete/incomplete market)**: A market is complete if a security of any value $X \in \mathbb{R}^m$ (row vector) at time t=1 is attainable. All other markets are incomplete.*

Proposition 1.8. *A market is complete if and only if the matrix $S_1 = \left(S_1^{(i)}(\omega_j) \right)_{(i,j)}$ has rank m.*

Proof. We have:

$$\begin{aligned} \text{The market is complete} &\Leftrightarrow \forall X \in \mathbb{R}^m,\ \exists \theta \in \mathbb{R}^{N+1},\ X = \theta\, S_1 \\ &\Leftrightarrow \forall X \in \mathbb{R}^m,\ \exists \theta \in \mathbb{R}^{N+1},\ X^T = S_1^T\, \theta^T \\ &\Leftrightarrow \operatorname{Im}\left(S_1^T\right) = \mathbb{R}^m \\ &\Leftrightarrow \dim\left(\operatorname{Im}\left(S_1^T\right)\right) = \operatorname{rank}\left(S_1^T\right) = m \\ &\Leftrightarrow \operatorname{rank}\left(S_1\right) = m. \end{aligned}$$

□

Remark 1.9. *It has been shown that the first example given in this section does not contain any arbitrage possibility. This market is incomplete, though, since matrix S_1 has rank at most 3, which is less than m (recall that, in this case, $m = 4$).*

Exercise: Consider the continued example addressed in this section. Does this example describe a complete or an incomplete market?

Definition 1.10. ***(link between arbitrage and equivalent portfolio)***: *A type A arbitrage is a portfolio $\theta \in \mathbb{R}^{N+1}$ (row vector) so that*

- $\theta\, S_0 < 0$,
- $\theta\, S_1 = 0$.

A type B arbitrage is a portfolio $\theta \in \mathbb{R}^{N+1}$ (row vector) so that

- $\theta\, S_0 \leq 0$,
- $\theta\, S_1 \geq 0$,
- $\theta\, S_1 \neq 0$.

Definition 1.11. ***(state price)***: *A column vector $(\pi_1, ..., \pi_m)^T > 0$ is a vector of state prices if the value $V \in \mathbb{R}$ at time $t = 0$ of any attainable security $X \in \mathbb{R}^m$ (row vector) at $t = 1$ reads:*

$$V = X\,(\pi_1, ..., \pi_m)^T = \sum_{j=1}^{m} \pi_j\, X(\omega_j)$$

where $X(\omega_j)$ is the value of this security at time $t = 1$ in state ω_j. Price π_j is referred to as the $j-th$ state price.

Remark 1.12. *If the $j-th$ elementary security is attainable, then its value at $t = 0$ is necessarily π_j.*

Proof. We have

$$e_j\,(\pi_1, ..., \pi_m)^T = (0, ..., 0, 1, 0, ..., 0)\,(\pi_1, ..., \pi_m)^T = \pi_j.$$

□

Proposition 1.13. *Assume that the linear pricing hypothesis is satisfied. Then, $(\pi_1, ..., \pi_m)^T \in \mathbb{R}^m$ (column vector) is a positive vector. Then,*

$$(\pi_1, ..., \pi_m)^T \text{ is a vector of state prices} \Leftrightarrow \begin{cases} S_1\,(\pi_1, ..., \pi_m)^T = S_0 = \left(S_0^{(0)}, ..., S_0^{(N)}\right)^T, \\ (\pi_1, ..., \pi_m) > 0. \end{cases}$$

Proof. Assume that $(\pi_1, ..., \pi_m)^T$ is a vector of state prices. We have directly $(\pi_1, ..., \pi_m) > 0$. We now prove that $S_1\,(\pi_1, ..., \pi_m)^T = S_0$. For all $\theta \in \mathbb{R}^{N+1}$ (row vector), $X = \theta\, S_1$ is an atteignable security, then, since $(\pi_1, ..., \pi_m)^T$ is a vector of state prices, the value V of this security X at time $t = 0$ is

$$V = X\,(\pi_1, ..., \pi_m)^T = \theta\, S_1\,(\pi_1, ..., \pi_m)^T.$$

However,

$$X = \theta\, S_1 \Leftrightarrow \forall j \in [\![1, m]\!], \quad X_j = X(\omega_j) = \sum_{i=0}^{N} \theta_i\, S_1^{(i)}(\omega_j),$$

then, under the linear pricing hypothesis, the value V of the security X at time $t = 0$ is

$$V = \sum_{i=0}^{N} \theta_i\, S_0^{(i)} = \theta\, S_0.$$

Then

$$V = \theta\, S_0 = \theta\, S_1\,(\pi_1, ..., \pi_m)^T \Rightarrow \theta\left(S_0 - S_1\,(\pi_1, ..., \pi_m)^T\right) = 0$$

and this for all $\theta \in \mathbb{R}^{N+1}$. We conclude that $S_1\,(\pi_1, ..., \pi_m)^T = S_0$.
Assume that $S_1\,(\pi_1, ..., \pi_m)^T = S_0$ with $(\pi_1, ..., \pi_m) > 0$ and prove that $(\pi_1, ..., \pi_m)^T$ is a vector of state prices. Set $X \in \mathbb{R}^m$ (row vector) an atteignable security at time $t = 1$. We want prove that under these hypothesis, the value $V \in \mathbb{R}$ of this atteignable security at time $t = 0$ is

$$V = X\,(\pi_1, ..., \pi_m)^T.$$

Since X is an atteignable security, there exists a portfolio $\theta \in \mathbb{R}^{N+1}$ (row vector) such that

$$X = \theta\, S_1 \Leftrightarrow \forall j \in [\![1, m]\!], \quad X_j = X(\omega_j) = \sum_{i=0}^{N} \theta_i\, S_1^{(i)}(\omega_j).$$

Under the linear pricing hypothesis, the value V of the security X at time $t = 0$ is

$$V = \sum_{i=0}^{N} \theta_i \, S_0^{(i)} = \theta \, S_0.$$

Since $S_1 \, (\pi_1, ..., \pi_m)^T = S_0$ and since $X = \theta \, S_1$, we have

$$V = \theta \, S_0 = \theta \, S_1 \, (\pi_1, ..., \pi_m)^T = X \, (\pi_1, ..., \pi_m)^T.$$

$\square$

Example (continued): Using the same example than above, one can check that (under the linear pricing hypothesis), $(0.2433, 0.1156, 0.3140, 0.3168)^T$ is a vector of state prices. Indeed, $(0.2433, 0.1156, 0.3140, 0.3168)^T \geq 0$ and

$$S_1 \, (0.2433, 0.1156, 0.3140, 0.3168)^T = \begin{pmatrix} 1.03 & 1.03 & 1.03 & 1.03 \\ 3 & 4 & 2 & 5 \\ 2 & 1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0.2433 \\ 0.1156 \\ 0.3140 \\ 0.3168 \end{pmatrix} = \begin{pmatrix} 1.0194 \\ 3.4045 \\ 2.4917 \end{pmatrix} = S_0.$$

Further observe that

$$(0, 0.3102, 0.4113, 0.2682)^T + \varepsilon \, (0.7372, -0.5898, -0.2949, 0.1474)^T$$

is also a vector of state prices for the real numbers ε so that all the entries of the vector are positive.

Method: In practice, if the linear pricing hypothesis holds, one can find a vector of state prices $(\pi_1, ..., \pi_m)^T \in \mathbb{R}^m$ by solving the following system

$$S_1 \, (\pi_1, ..., \pi_m)^T = S_0$$

and subsequently choosing a positive solution. Observe that such solution do not necessarily exist!

1.5 Deflating by the numeraire security

Definition 1.14. ***(risk-neutral security or numeraire)****: A numeraire is a security that produces a positive benefit at a positive rate r. An investment of 1 into such a numeraire security at time $t = 0$ will be worth $1 + r$ at time $t = 1$.*

Example (continued): the first security in the above example is a numeraire of rate $r = 0.0104$ Indeed, we have $S_0^{(0)} = 1.0194$ and $S_1^{(0)}(\omega_j) = 1.03$ for all $j \in [\![1, 4]\!]$. As $1.03 > 1.0194$ it is a numeraire and we have

$$(1 + r)S_0^{(0)} = S_1^{(0)}(\omega_j) \Leftrightarrow r = \frac{S_1^{(0)}(\omega_j)}{S_0^{(0)}} - 1 = 0.0104.$$

Notation: In the following of this text, it is assumed that the security 0 is a numeraire. In finance, it is usual and useful to express the values of the securities in a market by using some numeraire as a unit, which amounts to study the ratio

$$\frac{S_t^{(i)}(\omega_j)}{S_t{}^{(0)}(\omega_j)}$$

instead of $S_t^{(i)}(\omega_j)$. This convention is referred to as deflating a security by a numeraire. This models the fact that, because of the risk-neutral rate, the least profit that can be achieved without a risk is positive (think, for instance, of the inflation rate!).

1.6 Risk-neutral probability distribution

Definition 1.15. ***(risk-neutral distribution)****: The distribution $\mathcal{Q} = (q_1, ..., q_m) \in \mathbb{R}^m$ (of the states $\omega_1, ..., \omega_m$ at instant $t = 1$) is a risk-neutral probability distribution if:*

1. *$q_j > 0$ for all $j \in [\![1, m]\!]$,*

2. *$\sum_{j=1}^{m} q_j = 1$,*

3. *the deflated security prices are martingales. That is,*

$$\forall i \in [\![0, N]\!] : \quad \frac{S_0^{(i)}}{S_0^{(0)}} = \mathbb{E}^{\mathcal{Q}}\left(\frac{S_1^{(i)}}{S_1^{(0)}}\right)$$
$$= \sum_{j=1}^{m} q_j \frac{S_1^{(i)}(\omega_j)}{S_1^{(0)}(\omega_j)}.$$

2 Martingale pricing theory: single-period models

We are now ready to derive the main results of martingale pricing theory for single period models.

2.1 Link between no arbitrage, the existence of a state prices and the existence of a risk-neutral probability distribution

Proposition 2.1. ***(link between the existence of a risk-neutral distribution and the absence of arbitrage):*** *If there exists a risk-neutral probability distribution $\mathcal{Q} \in]0,1]^m$, then there is no possible arbitrage $\theta \in \mathbb{R}^{N+1}$ (row vector).*

Proof. Assume that we have vector $\mathcal{Q} \in]0,1]^m$ is a risk-neutral distribution. For a row vector $\theta \in \mathbb{R}^{N+1}$, we have

$$\theta\, S_0 = \sum_{i=0}^{N} \theta_i\, S_0^{(i)} \quad \text{and} \quad \forall j \in [\![1, m]\!] : \quad (\theta\, S_1)_j = \sum_{i=0}^{N} \theta_i\, S_1^{(i)}(\omega_j).$$

Since $\mathcal{Q}$ is a risk-neutral distribution, we have

$$\begin{aligned}
\theta\, S_0 &= \sum_{i=0}^{N} \theta_i\, S_0^{(i)} = \sum_{i=0}^{N} \theta_i\, S_0^{(0)} \mathbb{E}^{\mathcal{Q}}\left(\frac{S_1^{(i)}}{S_1^{(0)}}\right) \\
&= \sum_{i=0}^{N} \theta_i\, S_0^{(0)} \sum_{j=1}^{m} q_j \frac{S_1^{(i)}(\omega_j)}{S_1^{(0)}(\omega_j)} = \sum_{j=1}^{m} q_j\, \frac{S_0^{(0)}}{S_1^{(0)}(\omega_j)} \sum_{i=0}^{N} \theta_i S_1^{(i)}(\omega_j) \\
\Rightarrow \theta\, S_0 &= \sum_{j=1}^{m} q_j\, \frac{S_0^{(0)}}{S_1^{(0)}(\omega_j)}\, \left(\theta S_1\right)_j = \sum_{j=1}^{m} q_j\, \frac{S_0^{(0)}}{S_1^{(0)}(\omega_j)}\, \left(\theta S_1\right)_j. \qquad (1)
\end{aligned}$$

θ is a type A arbitrage if $\theta\, S_0 < 0$ and $\theta\, S_1 = 0$ that is not possible because (1) is satisfied. θ is a type B arbitrage if $\theta\, S_0 \leq 0$, $\theta\, S_1 \geq 0$ and $\theta\, S_1 \neq 0$ that is not possible because (1) is satisfied and $q_j\, \frac{S_0^{(0)}}{S_1^{(0)}(\omega_j)} > 0$ for all $j \in [\![1, m]\!]$. □

Theorem 2.2. ***(link between the existence of a state prices and the existence of a risk-neutral probability distribution):*** *If there exists a vector of state prices $(\pi_1, ..., \pi_m)^T$, then there exists a risk-neutral probability distribution $\mathcal{Q} \in]0,1]^m$.*

Proof. Assume that there exists a vector of state prices $(\pi_1, ..., \pi_m)^T$. We want prove that there exists $\mathcal{Q} = (q_1, ..., q_m) \in]0,1]^m$ such that

$$\begin{cases} \sum\limits_{j=1}^{m} q_j = 1, \\ \forall i \in [\![0, N]\!] : \quad \frac{S_0^{(i)}}{S_0^{(0)}} = \mathbb{E}^{\mathcal{Q}}\left(\frac{S_1^{(i)}}{S_1^{(0)}}\right) = \sum\limits_{j=1}^{m} q_j \frac{S_1^{(i)}(\omega_j)}{S_1^{(0)}(\omega_j)}. \end{cases}$$

Since $(\pi_1, ..., \pi_m)^T$ is a vector of state prices, for all atteignable security $X \in \mathbb{R}^m$ at $t = 1$, the value $V \in \mathbb{R}$ at $t = 0$ of this security is

$$V = X\, (\pi_1, ..., \pi_m)^T.$$

Since for all $i \in [\![0, N]\!]$, $S_1^{(i)} = \left(S_1^{(i)}(\omega_1), ..., S_1^{(i)}(\omega_m)\right) = (0, ..., 0, 1, 0, ..., 0)\; S_1$ is an atteignable security, the value V at time $t = 0$ of this security is

$$V = S_1^{(i)}(\pi_1, ..., \pi_m)^T.$$

However, the value at time $t = 0$ of the atteignable security $S_1^{(i)}$ is $S_0^{(i)}$, then $V = S_0^{(i)}$ and we have

$$\begin{aligned}\forall i \in [\![0, N]\!]: \quad S_0^{(i)} &= S_1^{(i)}(\pi_1, ..., \pi_m)^T = \sum_{j=1}^{m} S_1^{(i)}(\omega_j)\, \pi_j \\ &= \sum_{j=1}^{m} \left(\pi_j\, S_1^{(0)}(\omega_j)\right) \frac{S_1^{(i)}(\omega_j)}{S_1^{(0)}(\omega_j)}.\end{aligned}$$

Then,

$$\forall i \in [\![0, N]\!]: \quad \frac{S_0^{(i)}}{S_0^{(0)}} = \sum_{j=1}^{m} \left(\pi_j\, \frac{S_1^{(0)}(\omega_j)}{S_0^{(0)}}\right) \frac{S_1^{(i)}(\omega_j)}{S_1^{(0)}(\omega_j)}.$$

By setting $\mathcal{Q} = (q_1, ..., q_m)$ with $q_j = \pi_j\, \frac{S_1^{(0)}(\omega_j)}{S_0^{(0)}} > 0$, we obtain that $\mathcal{Q}$ is a risk-neutral probability distribution because

$$\sum_{j=1}^{m} q_j = \sum_{j=1}^{m} \pi_j\, \frac{S_1^{(0)}(\omega_j)}{S_0^{(0)}} = \frac{\sum_{j=1}^{m} \pi_j\, S_1^{(0)}(\omega_j)}{S_0^{(0)}} = \frac{\sum_{j=1}^{m} \pi_j\, S_1^{(0)}(\omega_j)}{\sum_{j=1}^{m} \pi_j\, S_1^{(0)}(\omega_j)} = 1.$$

□

Remark 2.3. *Probability distributions $\mathcal{P}$ and $\mathcal{Q}$ are distinct, but they are related (for instance, $p_j > 0$ if and only if $q_j > 0$). In practice, distribution $\mathcal{P}$ is most often unknown, so one will use distribution $\mathcal{Q}$ that can be (at least partially) computed and that has good properties.*

Theorem 2.4. ***(link between no arbitrage and the existence of a state prices):*** *When the model includes only one period of time (as done up to now), there is no possible arbitrage if and only if there exists a vector of state prices $(\pi_1, ..., \pi_m) > 0$.*

Proof. We have that

$$\begin{aligned}\text{There exists a vector of state prices } (\pi_1, ..., \pi_m) &\Rightarrow \text{There exists a risk-neutral distribution } \mathcal{Q} \in\,]0, +\infty[^m \\ &\Rightarrow \text{There is no possible arbitrage } \theta \in \mathbb{R}^{N+1} \text{ (row vector).}\end{aligned}$$

The reciprocal is more complicated. The proof relies on the following Lemma (this is a duality result in linear programming).

Lemma 2.5. *Let A be an $m \times n$ matrix and suppose that the matrix equation $A\, x = p$ for $p \geq 0$ cannot be solve except for the case $p = 0$. Then there exists a vector $y > 0$ such that $A^T\, y = 0$.*

Assume that there is no possible arbitrage $\theta \in \mathbb{R}^{N+1}$ (row vector). Consider the $(m+1) \times (N+1)$ matrix A, defined by

$$A = \begin{pmatrix} -S_0^T \\ S_1^T \end{pmatrix} = \begin{pmatrix} -S_0^{(0)} & \cdots & -S_0^{(N)} \\ S_1^{(0)}(\omega_1) & \cdots & S_1^{(N)}(\omega_1) \\ \vdots & \vdots & \vdots \\ S_1^{(0)}(\omega_m) & \cdots & S_1^{(N)}(\omega_m) \end{pmatrix}.$$

Moreover, we have

$$\theta \in \mathbb{R}^{N+1} \text{ is an arbitrage of type A} \Leftrightarrow \begin{cases} \theta\, S_0 < 0 \\ \theta\, S_1 \geq 0 \end{cases} \Leftrightarrow \begin{cases} -S_0^T \theta^T > 0 \\ S_1^T\, \theta^T \geq 0 \end{cases} \Leftrightarrow \begin{cases} -S_0^T \theta^T \geq 0 \\ S_1^T\, \theta^T \geq 0 \\ -S_0^T \theta^T \neq 0 \end{cases} \Leftrightarrow \begin{cases} A\theta^T \geq 0 \\ -S_0^T \theta^T \neq 0 \end{cases}$$

and

$$\theta \in \mathbb{R}^{N+1} \text{ is an arbitrage of type B} \Leftrightarrow \begin{cases} \theta\, S_0 \leq 0 \\ \theta\, S_1 \geq 0 \\ \theta\, S_1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} -S_0^T \theta^T \geq 0 \\ S_1^T\, \theta^T \geq 0 \\ S_1^T\, \theta^T \neq 0. \end{cases} \Leftrightarrow \begin{cases} -S_0^T \theta^T \geq 0 \\ S_1^T\, \theta^T \geq 0 \\ S_1^T\, \theta^T \neq 0. \end{cases} \Leftrightarrow \begin{cases} A\, \theta^T \geq 0 \\ S_1^T\, \theta^T \neq 0. \end{cases}$$

Since there is no possible arbitrage $\theta \in \mathbb{R}^{N+1}$, the system

$$\begin{cases} A\, \theta^T \geq 0 \\ A\, \theta^T \neq 0 \end{cases}$$

has no solution. The Lemma 2.5 gives us that there exists a vector $\pi = (\pi_0, \pi_1, ..., \pi_m)^T \in \mathbb{R}^{m+1}$ such that

$$\begin{cases} A^T\pi = 0 \\ \pi > 0 \end{cases} \Leftrightarrow \begin{cases} (-S_0, S_1)\begin{pmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_m \end{pmatrix} = 0 \\ (\pi_0, \pi_1, ..., \pi_m) > 0 \end{cases} \Leftrightarrow \begin{cases} \forall i \in [\![1, m]\!]; \quad \sum_{j=1}^{m} S_1^{(i)}(\omega_j)\pi_j = S_0^{(i)}\pi_0 \\ (\pi_0, \pi_1, ..., \pi_m) > 0. \end{cases}$$

We divide by $\pi_0 > 0$ and we obtain

$$\begin{cases} \forall i \in [\![1, m]\!]; \quad \sum_{j=1}^{m} S_1^{(i)}(\omega_j)\bar{\pi}_j = S_0^{(i)} \\ (\bar{\pi}_1, ..., \bar{\pi}_m) > 0 \end{cases}$$

where $\bar{\pi}_j = \frac{\pi_j}{\pi_0}$ for all $j \in [\![1, m]\!]$. Then, $S_1(\bar{\pi}_1, ..., \bar{\pi}_m)^T = S_0$ with $(\bar{\pi}_1, ..., \bar{\pi}_m)^T > 0$ and $(\bar{\pi}_1, ..., \bar{\pi}_m)^T$ is a vector of state prices. □

Example: In the above example, there is no possible arbitrage since there is a vector of state prices.

Theorem 2.6. ***(Fundamental theorem of asset pricing: part 1)****: Assume that the market contains a numeraire security. Then, the absence of arbitrage, the existence of a vector of state prices and the existence of a risk-neutral distribution are all equivalent.*

Proof. We have

$$\begin{aligned} \text{There exists a vector } (\pi_1, ..., \pi_m)^T \text{ of state prices} &\Rightarrow \text{There exists a risk-neutral distribution } Q \\ &\Rightarrow \text{There is no possible arbitrage} \\ &\Rightarrow \text{There exists a vector } (\pi_1, ..., \pi_m)^T \text{ of state prices.} \end{aligned}$$

□

Exercise: Consider a market with three securities with values $1.0, 1.9571$ and 2.2048 at time $t = 0$. Assume that there are two possible states ω_1 and ω_2 at time $t = 1$ so that:

$$\begin{aligned} S_1(\omega_1) &= (1.05, 2, 3)^T, \\ S_1(\omega_2) &= (1.05, 1, 2)^T. \end{aligned}$$

Further assume that the linear pricing hypothesis is satisfied. Show that there does not exist a vector of state prices for this model. Explain why this point explain the existence of an arbitrage situation within this market, and find an arbitrage portfolio θ.

Exercise: Consider the following market, with 4 states at $t = 1$ and 4 securities (therefore $m = 4$ and $N+1 = 4$)

$$\begin{aligned} S_0 &= {}^t(1.0194, 3.4045, 2.4917, 0.1548)^T, \\ S_1(\omega_1) &= (1.03, 3, 2, 1)^T, \\ S_1(\omega_2) &= (1.03, 4, 1, 2)^T, \\ S_1(\omega_3) &= (1.03, 2, 4, 1)^T, \\ S_1(\omega_4) &= (1.03, 5, 2, -2)^T. \end{aligned}$$

Show that matrix S_1 has rank 4, and that this market is therefore complete. Show that this market has no arbitrage.

Proposition 2.7. *If a market is incomplete, then at least one elementary security $e_j = (0, ..., 0, 1, 0, ..., 0)$ is not attainable.*

Proof. Assume that all $j \in [\![1, m]\!]$, e_j is atteignable. Then,

$$\forall j \in [\![1, m]\!], , \exists \theta^j \in \mathbb{R}^{N+1}, \quad e_j = \theta^j\, S_1.$$

Since the market is incomplete, we have

$$\exists X \in \mathbb{R}^m, \ \forall \theta \in \mathbb{R}^{N+1}, \quad X \neq \theta\, S_1.$$

However,

$$X=(x_1,...,x_m)=\sum_{j=1}^{m}x_j\ e_j=\left(\sum_{j=1}^{m}x_j\ \theta^j\right)\ S_1$$

which gives us a contradiction because X is not atteignable. □

Remark 2.8. *Assume that the market is incomplete, and therefore that, at least one elementary security, eay e_j, is not attainable. If there is not arbitrage situation within this market, it is still possible to find a vector of state price (because of the above theorem).*

2.2 Link between a complete market without arbitrage and the existence of a unique vector of state prices

Theorem 2.9. ***(Fundamental theorem of asset pricing: part 2)****: Consider a market without arbitrage and assume that this market contains a numeraire security. In this case, the market is complete if and only if there exists a unique risk-neutral distribution (or, equivalently, a unique vector of state prices).*

Proof. Assume that the market is complete. Since there is no arbitrage, we have the existence of a vector of state prices $(\pi_1,...,\pi_m)>0$ (see Theorem 2.6). Assume that $(\bar{\pi}_1,...,\bar{\pi}_m)>0$ is a vector of state prices and prove that $(\pi_1,...,\pi_m)=(\bar{\pi}_1,...,\bar{\pi}_m)$. Since $(\pi_1,...,\pi_m)$ and $(\bar{\pi}_1,...,\bar{\pi}_m)$ are vectors of state prices, we have

$$S_1\ (\pi_1,...,\pi_m)^T=S_0=S_1\ (\bar{\pi}_1,...,\bar{\pi}_m)^T\Rightarrow S_1\left[(\pi_1,...,\pi_m)^T-(\bar{\pi}_1,...,\bar{\pi}_m)^T\right]=0$$

Since the market is complete, $rank(S_1)=m$ and then $\dim\mathrm{Ker}(S_1)=\dim(\mathbb{R}^m)-rank(S_1)=m-m=0$ (S_1 is $(N+1)\times m$ matrix). Then, S_1 is injective and

$$S_1\left[(\pi_1,...,\pi_m)^T-(\bar{\pi}_1,...,\bar{\pi}_m)^T\right]=0\Rightarrow(\pi_1,...,\pi_m)^T=(\bar{\pi}_1,...,\bar{\pi}_m)^T.$$

Then, there exists a unique vector of states prices.
Assume now that there exists a unique vector of state prices $(\pi_1,...,\pi_m)>0$. If the market is not complete, we have $rank(S_1)<m$ (see Proposition 1.8) then since S_1 is $(N+1)\times m$ matrix, we have

$$\dim\mathrm{Ker}\,S_1=\dim(\mathbb{R}^m)-rank(S_1)=m-rank(S_1)>0$$

$$\Rightarrow\exists(h_1,...,h_m)\in\mathbb{R}^m,\quad\begin{cases}(h_1,...,h_m)\neq0,\\ S_1(h_1,...,h_m)^T=0.\end{cases}$$

Since $(\pi_1,...,\pi_m)>0$ is a vector of state prices, we have with the Proposition 1.13 (since the market is without arbitrage, the pricing is necessary linear) that

$$S_1(\pi_1,...,\pi_m)^T=S_0.$$

and then for all $\lambda>0$, we obtain

$$S_1\left[(\pi_1,...,\pi_m)^T-\lambda(h_1,...,h_m)^T\right]=S_0-0=S_0.$$

Since

$$(\pi_1,...,\pi_m)^T-\lambda(h_1,...,h_m)^T\underset{\lambda\to0}{\longrightarrow}(\pi_1,...,\pi_m)^T>0,$$

there exists a $\lambda>0$ such that $(\pi_1,...,\pi_m)^T-\lambda(h_1,...,h_m)^T>0$ and then $(\pi_1,...,\pi_m)^T-\lambda(h_1,...,h_m)^T$ is a vector of state prices different from $(\pi_1,...,\pi_m)^T$. It is a contradiction with the fact that $(\pi_1,...,\pi_m)^T$ is the unique vector of state prices and then the market is complete. □

3 Discrete pricing models over several periods

Notations: It will be assumed here that there are T consecutive periods of time (from $t=0$ to $t=T$). As in the single period case, the market is made up of $N+1$ securities. The value of $i-th$ security at time t is called $S_t^{(i)}$. Moreover, there are m possible states ω_1 to ω_m at time $t=T$. The probability that state ω_j occurs is denoted by p_j. The vector $P=(p_1,...,p_m)$ is therefore the probability distribution of these states.

Remark 3.1. *The state reached at a time $t<T$ determines a subset of the states $\omega_1,...,\omega_m$ that cannot be reached any more at time T.*

Remark 3.2. *A model over several periods is actually made up of several models over one period.*

3.1 Trading strategies and auto-financing trading strategies

Definition 3.3. ***(trading strategy)****: A trading strategy is a portfolio $\theta_t = \left(\theta_t^{(0)}(\omega), ..., \theta_t^{(N)}(\omega)\right)$ that gives the number of units of each security held just before trading at time t, as a function of t and ω.*

For example, $\theta_t^{(i)}(\omega)$ is the number of units of the i^{th} security held between times $t-1$ and t in state ω. We will sometimes write $\theta_t^{(i)}$, omitting the explicit dependance on ω.

Remark 3.4. *Vector θ_t is chosen at time $t-1$ and remained unchanged during the time interval $]t-1, t[$.*

Remark 3.5. *Vector θ_t being chosen at date $t-1$, it depends on the state reached at time $t-1$. In particular, this vector is dependent both on the state (therefore, random) and on the decisions made at each date. The rules used to make these decisions are called a strategy.*

Definition 3.6. *The value process $V_t(\theta)$ of a trading strategy θ_t at time t is the value of portfolio θ_t:*

$$V_t = \sum_{i=0}^{N} \theta_t^{(i)}\, S_t^{(i)}.$$

Definition 3.7. ***(auto-financed trading strategy)****: A trading strategy θ_t is called auto-financed if no amount of money is injected or withdrawn at any time $t > 0$ from the portfolio. Hence, the variations of V_t are only due to the fluctuations of the market, and the decisions to change the composition of the portfolio by reinvesting the value of the securities held into other securities. In particular, an auto-financed trading strategy satisfies*

$$\forall t \in [\![1, T-1]\!], \quad V_t = \sum_{i=0}^{N} \theta_{t+1}^{(i)}\, S_t^{(i)}.$$

Exercise: Show that if strategy θ_t is auto-financed, then the corresponding value V_t satisfies

$$V_{t+1} - V_t = \sum_{i=0}^{N} \theta_{t+1}^{(i)} \left(S_{t+1}^{(i)} - S_t^{(i)}\right).$$

3.2 Arbitrage

Definition 3.8. ***(arbitrage for models over several periods)****: A type A arbitrage is an auto-financed strategy θ_t so that*

- $V_0 = \theta_0\, S_0 < 0$,
- $V_T = \theta_T\, S_T = 0$.

A type B arbitrage is an auto-financed strategy θ_t so that

- $V_0 = \theta_0\, S_0 = 0$,
- $V_T = \theta_T\, S_T \geq 0$,
- $\mathbb{E}_0^{\mathcal{P}}(V_T) > 0$.

3.3 Attainability and complete markets

Definition 3.9. ***(attainable security):*** *A security of value X at time $t = 1$ (recall that X is a vector with m entries $X(\omega_1), ..., X(\omega_m)$ that are the values of the security at time $t = T$ in every possible state) is attainable if there exists an auto-financed trading strategy θ_t so that the value V_T satisfies $V_T (= \theta_T\, S_T) = X$.*

Definition 3.10. ***(complete/incomplete market)****: A market is complete if a security of any value at time $t = T$ is attainable. All other markets are called incomplete.*

3.4 Risk-neutral probability distribution

We assume again that we have a mind a specific numeraire security with price process $S_t^{(0)}$.

Definition 3.11. ***(risk-neutral probability distribution):*** *A risk-neutral probability distribution is a distribution $\mathcal{Q} = (q_1, ..., q_m)$ of the states at time $t = T$ so that:*

1. *$q_j > 0$ for all $j \in [\![1, m]\!]$,*
2. *$\sum\limits_{j=1}^{m} q_j = 1$,*
3. *the deflated security prices are martingales. That is, for all s, $t \geq 0$ and all $i \in [\![0, N]\!]$:*

$$\frac{S_t^{(i)}}{S_t^{(0)}} = \mathbb{E}_t^{\mathcal{Q}} \left(\frac{S_{t+s}^{(i)}}{S_{t+s}^{(0)}} \right),$$

where $\mathbb{E}_t^{\mathcal{Q}}$ is the expectancy computed using probability distribution $\mathcal{Q} = (q_1, ..., q_m)$ conditionned to the information available at time t (and not $t + s$).

Natations: We note $\bar{V}_t$ the deflated value process associated to a value process V_t and $\bar{S}_t$ the deflated security prices. It means that

$$\forall i \in [\![0, N]\!], \quad \bar{S}_t^{(i)} := \frac{S_t^{(i)}}{S_t^{(0)}} \quad \text{and} \quad \bar{V}_t := \frac{V_t}{S_t^{(0)}} = \sum_{i=0}^{N} \theta_t^{(i)} \, \bar{S}_t^{(i)}.$$

4 Martingale pricing theory: multi-period models

We will now generalize the results for single-period models to multi-period models. This is easily done using our single-period results and requires very little extra work.

4.1 Link between no arbitrage and the existence of a risk-neutral probability distribution

Proposition 4.1. *If there exists a risk-neutral probability distribution $\mathcal{Q}$, then the deflated value process $\bar{V}_t = \frac{V_t}{S_t^{(0)}}$ is a $\mathcal{Q}$-martingale for any auto-financed trading strategy θ_t.*

Proof. Assume that there exists a risk-neutral probability distribution $\mathcal{Q}$. Let θ_t an auto-financed strategy and let $\bar{V}_{t+1} := \frac{V_{t+1}}{S_{t+1}^{(0)}}$ denote the deflated value process. Then, we have for all $t \geq 0$

$$\begin{aligned}
\mathbb{E}_t^{\mathcal{Q}} \left(\bar{V}_{t+1} \right) = \mathbb{E}_t^{\mathcal{Q}} \left(\frac{V_{t+1}}{S_{t+1}^{(0)}} \right) &= \mathbb{E}_t^{\mathcal{Q}} \left(\frac{\sum\limits_{i=0}^{N} \theta_{t+1}^{(i)} \, S_{t+1}^{(i)}}{S_{t+1}^{(0)}} \right) = \mathbb{E}_t^{\mathcal{Q}} \left(\sum_{i=0}^{N} \theta_{t+1}^{(i)} \, \bar{S}_{t+1}^{(i)} \right) \\
&= \sum_{i=0}^{N} \theta_{t+1}^{(i)} \, \mathbb{E}_t^{\mathcal{Q}} \left(\bar{S}_{t+1}^{(i)} \right) = \sum_{i=0}^{N} \theta_{t+1}^{(i)} \, \bar{S}_t^{(i)} \\
&= \bar{V}_t
\end{aligned}$$

demonstrating that $\bar{V}_t$ is indeed a $\mathcal{Q}$-martingale. ☐

Remark 4.2. *Note that the Proposition 4.1 implies that the deflated price of any atteignable security can be computed as the $\mathcal{Q}$-expectation of the terminal deflated value of the security*

$$\bar{V}_0 = \mathbb{E}_0^{\mathcal{Q}} \left(\bar{V}_T \right).$$

Theorem 4.3. ***(Fundamental theorem of asset pricing: Part 1):*** *In the model over several periods, there is no possible arbitrage situation if and only if there exists a risk-neutral probability distribution $\mathcal{Q}$.*

Proof. Assume that there exists a risk-neutral probability distribution $\mathcal{Q}$. With the Proposition 4.1, we obtain that

$$\bar{V}_0 = \mathbb{E}_0^{\mathcal{Q}}\left(\bar{V}_T\right).$$

Then, a A type arbitrage is not possible because

$$V_T = 0 \implies \bar{V}_T = 0 \implies \bar{V}_0 = \mathbb{E}_0^{\mathcal{Q}}\left(\bar{V}_T\right) = 0 \implies V_0 = 0.$$

and a B type arbitrage is not possible because

$$\mathbb{E}_0^{\mathcal{Q}}\left(V_T\right) > 0 \implies \mathbb{E}_0^{\mathcal{Q}}\left(\bar{V}_T\right) > 0 \implies \bar{V}_0 = \mathbb{E}_0^{\mathcal{Q}}\left(\bar{V}_T\right) > 0 \implies V_0 > 0.$$

Then, there is no possible arbitrage.
Assume that there is no possible arbitrage. Then, we can easily argue that there is no arbitrage (as defined by Definition 1.10) in any of the embedded one-periods models. Then, Theorem 2.6 implies that each of the embedded one-period models has a set $\mathcal{Q}_t$ of risk neutral probabilities. By multiplying these probabilities, we can construct an risk neutral probability distribution $\mathcal{Q}$ as defined in Definition 3.11 □

4.2 Link between a complete market without arbitrage and the existence of a unique risk-neutral probability distribution

Proposition 4.4. *In the model over several periods, the market is complete if and only if every model over one period it is made up of are complete.*

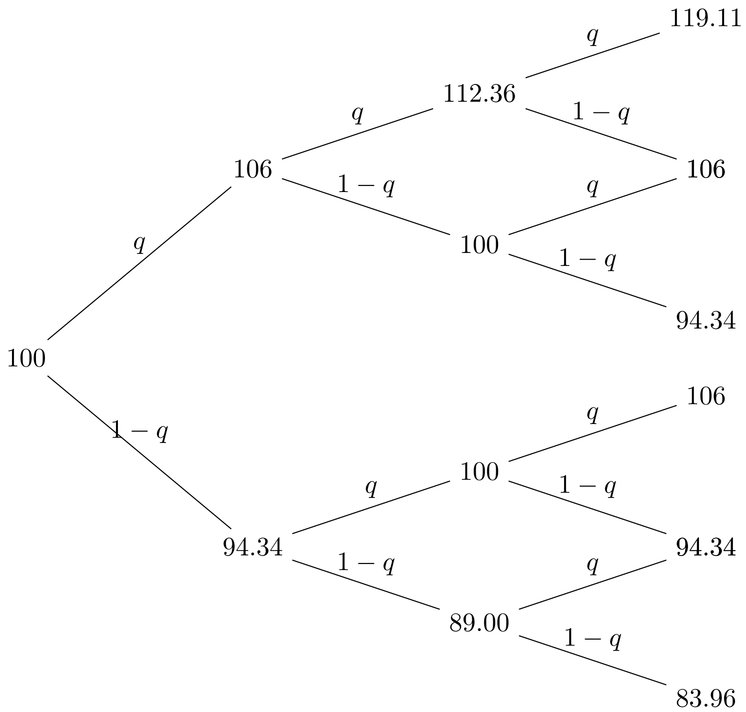
Theorem 4.5. ***(Fundamental theorem of asset pricing: part 2)**: Consider a model over several periods of some market. Assume that this market is without arbitrage and contains a numeraire security. In this case, the market is complete if and only if there exists a unique risk-neutral distribution $\mathcal{Q}$.*

Proof. Assume that the market is complete. Then by Proposition 4.4 every embedded one-period model is complete, then we can apply Theorem 2.9 to show that a unique risk-neutral distribution $\mathcal{Q}$ exists (which must exists since there is no arbitrage).
Assume now that $\mathcal{Q}$ is unique. Then, the risk-neutral probability measure corresponding to each one period model is also unique. Now apply Theorem 2.9 again to obtain that the multi-period model is complete. □

5 The binomial model

The binomial model is a particular model over several periods that describes the evolution with time of the price of a single security. If the price of the security is S at a given node (i.e. state) of this model, then only two outcomes are possible: at the next time, the price of the security increaes to $u \times S$, where $u > 1$ or decreases to $d \times S$ where $d = 1/u$. It is also assumed that there is one numeraire security of fixed risk-free rate r. These two securities (the risked security, or the numeraire) can be bought or sold "short" (i.e. borrowed and then immediately sold). $B_t = (1+r)^t$ is the value at time t of 1$ invested in the numeraire at time 0. One can see that all the sub-models over one period are identical (except for the price of the risked security).

Example: If at $t = 0$, the price of the risked security is 100 and $u = 1.06$, then the price of the risked security evolves as follows:



Proposition 5.1. *If there is no arbitrage in a binomial model then $d < 1 + r < u$.*

Proof. Assume that there is no arbitrage in a binomial model. The first fundamental theorem implies that there exists a risk-neutral probability distribution $\mathcal{Q} = (q, 1-q)$ satisfying $0 < q < 1$ and

$$\begin{aligned} \forall t \geq 0, \quad \frac{S_t}{S_t^{(0)}} &= \mathbb{E}_t^{\mathcal{Q}}\left(\frac{S_{t+1}}{S_{t+1}^{(0)}}\right) \\ &= q\frac{u\,S_t}{S_{t+1}^{(0)}} + (1-q)\frac{d\,S_t}{S_{t+1}^{(0)}} \end{aligned}$$

where $S_t^{(0)} = (1+r)^t$, then

$$\begin{aligned} S_t &= q\frac{u\,S_t}{1+r} + (1-q)\frac{d\,S_t}{1+r} \\ \Longrightarrow 1 + r &= qu + (1-q)d = q\,(u-d) + d \\ \Longrightarrow q &= \frac{1+r-d}{u-d}. \end{aligned}$$

Since $0 < q < 1$, we obtain $d < 1 + r < u$. □

Proposition 5.2. *If $d < 1+r < u$, a market modeled by a binomial model has no arbitrage and is also complete.*

Proof. Assume that $d < 1 + r < u$. The market is modeled by a binomial model, then

$$S_0 = \begin{pmatrix} S_0^{(0)} \\ S_0^{(1)} \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} (1+r)S_0^{(0)} & (1+r)S_0^{(0)} \\ uS_0^{(1)} & dS_0^{(1)} \end{pmatrix}.$$

By setting $q = \frac{1+r-d}{u-d}$, we have that $\mathcal{Q} = (q, 1-q)$ is a no-risk probability distribution. Indeed, $0 < q < 1$, $q + 1 - q = 1$ and

$$\begin{aligned} \mathbb{E}^{\mathcal{Q}}\left(\frac{S_1^{(0)}}{S_1^{(0)}}\right) &= 1 = \frac{S_0^{(0)}}{S_0^{(0)}} \\ \mathbb{E}^{\mathcal{Q}}\left(\frac{S_1^{(1)}}{S_1^{(0)}}\right) &= q\frac{S_1^{(1)}(\omega_1)}{S_1^{(0)}(\omega_1)} + (1-q)\frac{S_1^{(1)}(\omega_2)}{S_1^{(0)}(\omega_2)} \\ &= q\frac{u\,S_0^{(1)}}{(1+r)\,S_0^{(0)}} + (1-q)\frac{d\,S_0^{(1)}}{(1+r)\,S_0^{(0)}} = \frac{q\,u + (1-q)\,d}{1+r}\frac{S_0^{(1)}}{S_0^{(0)}} \\ &= \frac{q\,(u-d) + d}{1+r}\frac{S_0^{(1)}}{S_0^{(0)}} = \frac{S_0^{(1)}}{S_0^{(0)}} \end{aligned}$$

The market is also complete because the matrix

$$S_1 = \begin{pmatrix} (1+r)S_0^{(0)} & (1+r)S_0^{(0)} \\ uS_0^{(1)} & dS_0^{(1)} \end{pmatrix}$$

has rank $2 = m$. □

If $d < 1 + r < u$, the market modeled by a binomial model does not have arbitrage and is also complete. As a consequence, if X_T is the value at time T of a security, then its value X_t at time t is:

$$\frac{X_t}{(1+r)^t} = \mathbb{E}_t^{\mathbb{Q}}\left(\frac{X_T}{(1+r)^T}\right)$$
$$\Longrightarrow X_t = \frac{\mathbb{E}_t^{\mathbb{Q}}(X_T)}{(1+r)^{T-t}}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ is the expectancy computed using the risque-neutral probability distribution $q_1, ..., q_m$ conditionally to the information available to time t. The binomial model is therefore recombining, that is an increase followed by a decrease results in the same value than a decrease followed by an increase. In particular, whereas a model over several periods can have an exponentional number of nodes (as a function of the number of periods), the binomial model has only a quadratic number of nodes, which makes it much more computer-friendly.

6 Call and put

Definition 6.1. ***(call)****: A call option is a security that gives its owner the right to buy a given security (called the "underlying" security) at a date t>0 (exercise date) for a certain price (called "strike") determined at time $t = 0$.*

Definition 6.2. ***(European call)****: A european call option is a call whose exercise date $t = T > 0$ is determined at time $t = 0$. The date T is also called expiration date.*

Definition 6.3. ***(American call)****: An american call option is a call whose exercise date t can be any date from $t = 0$ to $T > 0$. The date T is fixed at time $t = 0$ and called expiration date.*

Example (call): Assume I buy 100 european call options over a given underlying security, with strike $K = 55$€ and expiration date december 31 this year. Assume that, today, the underlying securityis worth 47€. Every call option costs 5€(So it costs 500€ to buy 100 of them).

- If at expiration date, the underlying security costs 65€, I choose to exercice my options, and I buy 100 underlying securities for 55€ each, which I immediately sell for 65€ each. I have a profit of 10€ per security, that is 1000, to which I must subtract the price of my calls (500€).
- If at expiration date, the underlying security costs 50€, I choose not to exercise my options (otherwise I yould buy 100 securities for 5€ more than their actual price). I lose the price of my calls, that is 500€.

Definition 6.4. ***(put)*** *: A put option is a security that gives its owner the right to sell a given security (called the "underlying" security) at a date date t>0 (exercise date) for a certain price (called "strike") determined at time $t = 0$.*

Definition 6.5. ***(European put)****: A european put option is a put whose exercise date $t = T > 0$ is determined at time $t = 0$. The date T is also called expiration date.*

Definition 6.6. ***(American put)****: An american put option is a put whose exercise date t can be any date from $t = 0$ to $T > 0$. The date T is fixed at time $t = 0$ and called expiration date.*

Example (put): Assume I buy 100 european put options over a given underlying security, with strike $K = 55$€ and expiration date december 31 this year. Assume that, today, the underlying security is worth 47€. Every call option costs 5€ (So it costs 500€ to buy 100 of them).

- If at expiration date, the underlying security costs 65€, I choose not to exercise my options (otherwise I would sell 100 securities for 10€less than their actual price). I lose the price of my puts, that is 500€.

- If at expiration date, the underlying security costs 40€, I choose to exercice my options: I buy 100 underlying securities at the market price (40€) and I immediately sell them for 55€ each. I have a profit of 15€ per security, that is 1500€, to which I must subtract the price of my puts (500€).

Method (pricing of an option in the binomial model) :

1. Compute the value of the underlying security for every node of the model (from $t = 0$ to $t = T$, that is, from left to right in the tree),
2. Compute the risk-neutral probability distribution $(q, 1-q)$ for the (unique) model over one period constituting the binomial model (under the assumption that all deflated prices at martingales),
3. Compute the payoff at the exercise of the option at time T. These payoffs are the actual values (hence their prices) of the option at time T,
4. Compute the value of the option for every node of the model (from $t = T$ to $t = 0$, that is, from right to left in the tree) under the assumption that all deflated prices ar martingales,
5. The price of the option is the value obtained at time $t = 0$.

Exercise (pricing a european put): Compute the price of a european put option whose underlying security is the one given in the tree above, with expiration date $T = 3$, strike $K = 95$. It will be assumed that $r = 0.02$.

Exercise (pricing a european call): Compute the price of a european call over the same underlying security, and under the same conditions, but with $r = 0.04$.

Optimal strategy for american options: In each date t from 0 to T, one can choose to exercise the option or not. Obviously, this choice depends on the payoff found in each case. In practice, at time t and in a given state ω_j, the value of our option is the largest of the two following values:

- If one chooses to exercise, the payoff is K - S where S is the value of the underlying security at time t in state ω_j. K - S is therefore the value of our option under the assumption that we exercise it.
- If one chooses not to exercise, the payoff is computed from the values of the option at time $t+1$, assuming these values follow a martingale for the risk-neutral distribution.

Exercise: Compute the price of an american put option whose underlying security is the one given in the tree above, with expiration date $T = 3$, strike $K = 95$. It will be assumed that $r = 0.02$.

Exercise: Compute the price of an american call over the same underlying security, and under the same conditions, but with $K = 110$ and $r = 0.005$.