A low Mach correction for the Godunov scheme applied to the linear wave equation with porosity

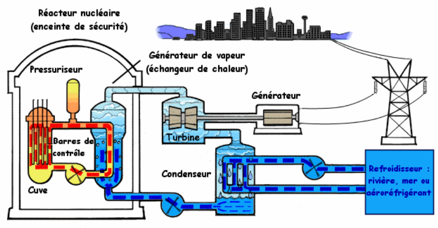
Jonathan Jung
Université de Pau et des Pays de l'Adour
& INRIA Bordeaux Sud Ouest, Cagire Team

Collaborators :
Stéphane Dellacherie (CEA, Saclay & LRC-Manon)
Pascal Omnes (CEA, Saclay & Université Paris 13)

Nantes, 26 November 2015

Study case :

- Nuclear core reactor.



Properties of the flow :

- Low Mach flow :

$$\begin{cases} |\mathbf{u}| \approx 5\ m.s^{-1}, \\ c \approx 500\ m.s^{-1}, \end{cases}$$

$$\Rightarrow M := \frac{|\mathbf{u}|}{c} \approx 10^{-2} \ll 1.$$

- Flow with variable cross-section (porosity).
- Compressible flow : shock wave in some accidental cases.

Aim :

- Develop a "compressible" numerical scheme that is accurate at low Mach number.

Barotropic Euler equations with porosity

- Barotropic Euler equations with porosity

$$\begin{cases} \partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \alpha\nabla p = 0, \end{cases}$$

  where $\alpha \in [\alpha_{\min}, 1]$ is the porosity with $\alpha_{\min} > 0$.

- Hyperbolic system (under the condition $p'(\rho) > 0$) with source term.
- Dimensionless : we introduce $\tilde{x} = \frac{x}{L}$, $\tilde{y} = \frac{y}{L}$, $\tilde{t} = \frac{t}{T}$, $\tilde{\alpha} = \frac{\alpha}{\alpha_0}$, $\tilde{\rho} = \frac{\rho}{\rho_0}$, $\tilde{u_x} = \frac{u_x}{u_0}$, $\tilde{u_y} = \frac{u_y}{u_0}$, $\tilde{p} = \frac{p}{p_0}$ avec $u_0 = \frac{L}{T}$, we obtain

$$\begin{cases} \partial_{\tilde{t}}(\tilde{\alpha}\tilde{\rho}) + \tilde{\nabla}\cdot(\tilde{\alpha}\tilde{\rho}\tilde{\mathbf{u}}) = 0, \\ \partial_{\tilde{t}}(\tilde{\alpha}\tilde{\rho}\tilde{\mathbf{u}}) + \tilde{\nabla}\cdot(\tilde{\alpha}\tilde{\rho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}) + \frac{\tilde{\alpha}}{M^2}\nabla\tilde{p} = 0, \end{cases} \quad \text{with } M = \frac{u_0}{c_0}.$$

Linear wave equation with porosity

- Change of variable $\tilde{\rho} := \tilde{\rho}_\star \left(1 + \frac{M}{a_\star}\tilde{r}\right)$, with $\begin{cases} a_\star^2 = \tilde{p}'(\tilde{\rho}_\star) \\ \frac{M}{a_\star}\tilde{r} \ll 1. \end{cases}$

$$\begin{cases} \partial_t(\tilde{\alpha}\tilde{r}) + \tilde{\nabla}\cdot(\tilde{\alpha}\tilde{r}\tilde{\mathbf{u}}) + \frac{a_\star}{M}\tilde{\nabla}\cdot(\tilde{\alpha}\tilde{\mathbf{u}}) = 0, \\ \partial_{\tilde{t}}(\tilde{\alpha}\tilde{\mathbf{u}}) + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})(\tilde{\alpha}\tilde{\mathbf{u}}) + \frac{\tilde{\alpha}}{M}\frac{\tilde{p}'\left(\tilde{\rho}_\star\left(1+\frac{M}{a_\star}\tilde{r}\right)\right)}{a_\star\left(1+\frac{M}{a_\star}\tilde{r}\right)}\nabla\tilde{r} = 0. \end{cases}$$

- Linearization around $(\tilde{r} = 0, \tilde{\mathbf{u}} = 0)$: **linear wave equation with porosity**

$$\begin{cases} \partial_t(\alpha r) + \frac{a_\star}{M}\nabla\cdot(\alpha\mathbf{u}) = 0, \\ \partial_t(\alpha\mathbf{u}) + \frac{a_\star}{M}\alpha\nabla r = 0. \end{cases}$$

- Kernel of the spatial operator : **incompressible space**

$$\mathcal{E}_\alpha := \left\{ q = (r,\mathbf{u})^T \in L^2_\alpha(\mathbb{T})^3 \,\middle|\, \nabla r = 0 \text{ and } \nabla\cdot(\alpha\mathbf{u}) = 0 \right\}.$$

Aim :

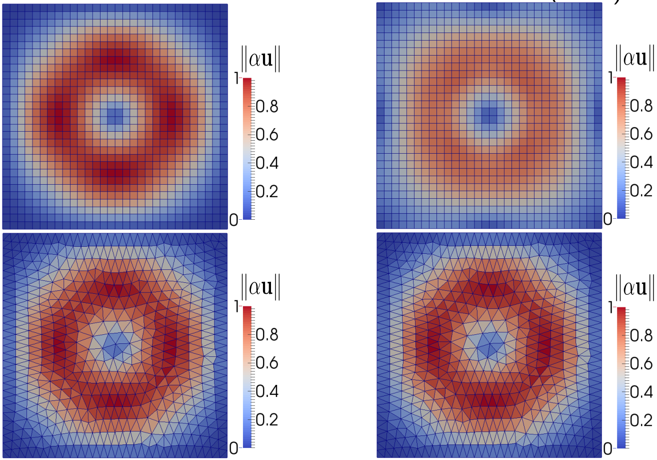
- Study the behavior of the numerical scheme with the incompressibles states $q \in \mathcal{E}_\alpha$.

Numerical problem : an initial incompressible condition $q_0 \in \mathcal{E}_\alpha$.

Initial condition : | At $t = 0.0001(\approx M)$:

Cartesian mesh

Triangular mesh



Aims :

- Found the origin of the problem on a cartesian mesh.
- Understand the triangular case.

Finite volume scheme

Godunov scheme on a triangular or cartesian mesh $(\Omega_i)_{1\leq i\leq N}$

$$\begin{cases} \frac{d}{dt}(\alpha r)_i + \frac{a_\star}{2M}\frac{1}{|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}|\left(\alpha\mathbf{u}\cdot\mathbf{n}\right)_{ij} = 0, \\ \frac{d}{dt}(\alpha\mathbf{u})_i + \frac{a_\star}{2M}\frac{\alpha_i}{|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| r_{ij}\mathbf{n}_{ij} = 0, \end{cases}$$

where $\left(r_{ij}, (\alpha\mathbf{u}\cdot\mathbf{n})_{ij}\right)$ is the solution of the $1D$ Riemann problem [1] in the direction $\mathbf{n}_{ij}$ on $\xi/t = 0$

$$\begin{cases} \alpha_{ij}\partial_t r_\xi + \frac{a_\star}{M}\partial_\xi\left((\alpha u)_\xi\right) = 0, \\ \partial_t\left((\alpha u)_\xi\right) + \frac{a_\star}{M}\alpha_{ij}\partial_\xi r_\xi = 0, \\ \left(r_\xi, (\alpha u)_\xi\right)(t=0,\xi) = \begin{cases} \left(r_i, (\alpha\mathbf{u})_i\cdot\mathbf{n}_{ij}\right) & \text{if } \xi < 0, \\ \left(r_j, (\alpha\mathbf{u})_j\cdot\mathbf{n}_{ij}\right) & \text{otherwise.} \end{cases} \end{cases}$$

1. S. Dellacherie, P. Omnes, On the Godunov scheme applied to the variable cross-section linear equation. FVCA6, (4) :313–321, 2011.

Numerical scheme

- The solution of the Riemann problem on $\xi/t = 0$ is given by

$$\begin{cases} r_{ij} = \dfrac{r_i + r_j}{2} + \dfrac{1}{2\alpha_{ij}}\big((\alpha\mathbf{u})_i - (\alpha\mathbf{u})_j\big)\cdot\mathbf{n}_{ij}, \\ \big((\alpha\mathbf{u})\cdot\mathbf{n}\big)_{ij} = \dfrac{\big((\alpha\mathbf{u})_i + (\alpha\mathbf{u})_j\big)\cdot\mathbf{n}_{ij}}{2} + \dfrac{\alpha_{ij}}{2}(r_i - r_j). \end{cases}$$

- Godunov scheme on a triangular or cartesian mesh $(\Omega_i)_{1\le i\le N}$ [DO11]

$$\begin{cases} \dfrac{d}{dt}(\alpha r)_i + \dfrac{a_\star}{2M}\dfrac{1}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\Big[\big((\alpha\mathbf{u})_i + (\alpha\mathbf{u})_j\big)\cdot\mathbf{n}_{ij} + \alpha_{ij}(r_i - r_j)\Big] = 0, \\ \dfrac{d}{dt}(\alpha\mathbf{u})_i + \dfrac{a_\star}{2M}\dfrac{\alpha_i}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\Big[r_i + r_j + \dfrac{\kappa}{\alpha_{ij}}\big((\alpha\mathbf{u})_i - (\alpha\mathbf{u})_j\big)\cdot\mathbf{n}_{ij}\Big]\mathbf{n}_{ij} = 0 \end{cases}$$

with $\kappa = 1$.

Found the origin of the problem on a cartesian mesh

Modified equation on a cartesian mesh :

- The Godunov scheme on a cartesian mesh $\Omega_{i,j}$ for αr gives us

$$\partial_t(\alpha r)_{i,j} + \frac{a_\star}{M}\frac{(\alpha u_x)_{i+1,j} - (\alpha u_x)_{i-1,j}}{2\Delta x} + \frac{a_\star}{M}\frac{(\alpha u_y)_{i,j+1} - (\alpha u_y)_{i,j-1}}{2\Delta y}$$
$$= \frac{a_\star}{2M\Delta x}\left(\alpha_{i+\frac{1}{2},j}\left(r_{i+1,j} - r_{i,j}\right) - \alpha_{i-\frac{1}{2},j}\left(r_{i,j} - r_{i-1,j}\right)\right)$$
$$+ \frac{a_\star}{2M\Delta y}\left(\alpha_{i,j+\frac{1}{2}}\left(r_{i,j+1} - r_{i,j}\right) - \alpha_{i,j-\frac{1}{2}}\left(r_{i,j} - r_{i,j-1}\right)\right).$$

- Then, the first order modified equation for αr is

$$\partial_t(\alpha r) + \frac{a_\star}{M}\nabla\cdot(\alpha\mathbf{u}) = \frac{a_\star\Delta x}{2M}\partial_x(\alpha\partial_x r) + \frac{a_\star\Delta y}{2M}\partial_y(\alpha\partial_y r).$$

Modified equation on a cartesian mesh

- We use the same method for $\alpha\mathbf{u}$ and we obtain the modified system

$$\partial_t(\alpha r) + \frac{a_\star}{M}\nabla\cdot(\alpha\mathbf{u}) - \frac{a_\star\Delta x}{2M}\partial_x(\alpha\partial_x r) - \frac{a_\star\Delta y}{2M}\partial_y(\alpha\partial_y r) = 0$$

$$\partial_t(\alpha\mathbf{u}) + \frac{a_\star}{M}\alpha\nabla r - \begin{pmatrix} \kappa\alpha\frac{a_\star\Delta x}{2M}\partial_x\left(\frac{1}{\alpha}\partial_x(\alpha u_x)\right) \\ \kappa\alpha\frac{a_\star\Delta y}{2M}\partial_y\left(\frac{1}{\alpha}\partial_y(\alpha u_y)\right) \end{pmatrix} = 0$$

with $\kappa = 1$. We write it as

$$\partial_t(\alpha q) + \frac{\mathcal{L}_{\kappa,\alpha}}{M}(q) = 0 \quad \text{with} \quad \mathcal{L}_{\kappa,\alpha} = L_\alpha - MB_{\kappa,\alpha}.$$

- **What is the relation between Ker $\mathcal{L}_{\kappa,\alpha}$ and $\mathcal{E}_\alpha$?**

Kernel of the modified equation

Proposition

1. *If $\kappa > 0$, we have*

$$Ker\,\mathcal{L}_{\kappa>0,\alpha} = \left\{q := (r, \boldsymbol{u})^T \middle| \nabla r = 0 \text{ and } \partial_x(\alpha u_x) = \partial_y(\alpha u_y) = 0\right\} \subsetneq \mathcal{E}_\alpha.$$

2. *If $\kappa = 0$, we have $Ker\,\mathcal{L}_{\kappa=0,\alpha} = \mathcal{E}_\alpha$.*

Conclusion of the study of the continuous case :

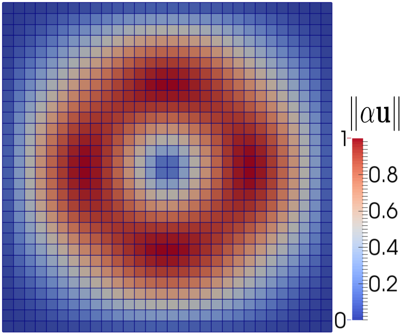
- Substitute $\kappa = 1$ by $\kappa = 0$ seams to allow to the Godunov scheme to preserve the incompressible states $q^0 \in \mathcal{E}_\alpha$ on cartesian meshes.

To do :

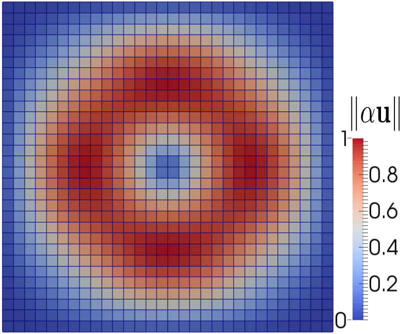
- Test this correction $(\kappa = 0)$ at the discret level.

Test of the low Mach correction $\kappa = 0$

- Initial condition $q^0 \in \mathcal{E}$:



- At $t = 0.0001 (= M)$:



Aims :

- Study the problem at the discrete level.
- Study the case of a triangular mesh.

Discrete study

- We define the spaces $\mathcal{E}_\alpha^\triangle$ and $\mathcal{E}_\alpha^\square$ associated to the incompressible space $\mathcal{E}_\alpha$ on a triangular and a cartesian mesh.
- Recall the Godunov scheme on a triangular or cartesian mesh $(\Omega_i)_{1\le i\le N}$ [DO11]

$$\begin{cases} \dfrac{d}{dt}(\alpha r)_i + \dfrac{a_\star}{2M}\dfrac{1}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\Big[\big((\alpha\mathbf{u})_i+(\alpha\mathbf{u})_j\big)\cdot\mathbf{n}_{ij}+\alpha_{ij}(r_i-r_j)\Big]=0, \\ \dfrac{d}{dt}(\alpha\mathbf{u})_i + \dfrac{a_\star}{2M}\dfrac{\alpha_i}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\Big[r_i+r_j+\dfrac{\kappa}{\alpha_{ij}}\big((\alpha\mathbf{u})_i-(\alpha\mathbf{u})_j\big)\cdot\mathbf{n}_{ij}\Big]\mathbf{n}_{ij}=0 \end{cases}$$

with $\kappa = 1$.

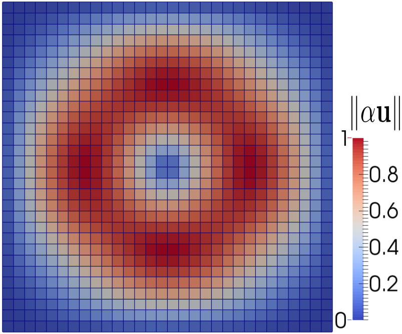
- We write it as

$$\frac{d}{dt}(\alpha q_h) + \frac{\mathbb{L}^h_{\kappa,\alpha}}{M}(q_h) = 0,$$

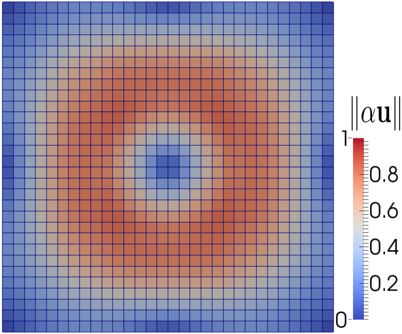
where $q_h = (r_i, \mathbf{u}_i)^T$.

Gudunov scheme $(\kappa = 1)$ on a cartesian mesh :

- Initial condition $q^0 \in \mathcal{E}_\alpha^\square$:



- At $t = 0.0001 (= M)$:

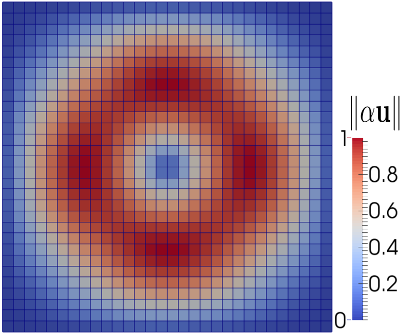


Proposition $(\kappa = 1$ on $\square)$

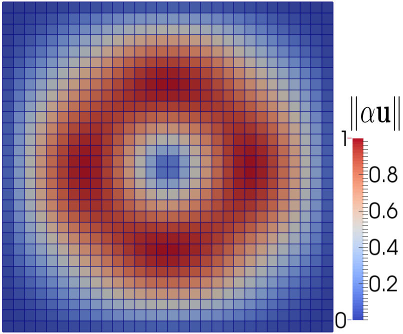
$$Ker\, \mathbb{L}^h_{\kappa=1,\alpha} \subsetneq \mathcal{E}_\alpha^\square .$$

Low Mach scheme $(\kappa = 0)$ on a cartesian mesh :

- Initial condition $q^0 \in \mathcal{E}^{\square}$:



- At $t = 0.0001 (= M)$:

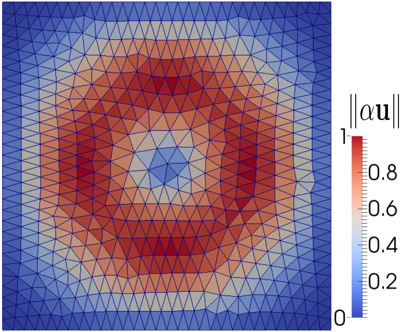


Proposition ($\kappa = 0$ on $\square$)

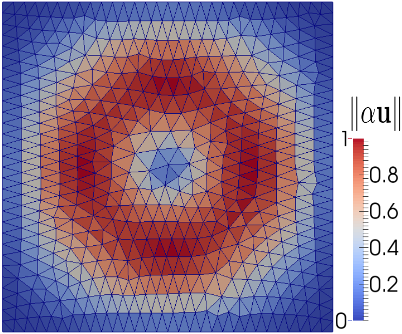
$$Ker\, \mathbb{L}^h_{\kappa=0,\alpha} = \mathcal{E}^{\square}_{\alpha}.$$

Godunov scheme $(\kappa = 1)$ on a triangular mesh :

- Initial condition $q^0 \in \mathcal{E}^{\triangle}$:



- At $t = 0.0001 (= M)$:



Proposition ($\kappa = 1$ on $\triangle$)

$$Ker\,\mathbb{L}^h_{\kappa=1,\alpha} = \mathcal{E}^{\triangle}_{\alpha}.$$

Conclusion on the discrete study of the kernel of the Godunov scheme

Conclusion :

- The Godunov scheme ($\kappa = 1$) does not preserve some incompressible states $\mathcal{E}_\alpha^\square$ on a cartesian mesh.
- The low Mach scheme ($\kappa = 0$) preserves the incompressible states $\mathcal{E}_\alpha^\square$ on a cartesian mesh.
- The Godunov scheme ($\kappa = 1$) preserves the incompressible states $\mathcal{E}_\alpha^\triangle$ on a triangular mesh.

BUT :

- We wish a correction that allows to obtain the Godunov scheme when the Mach number tends to 1.
- What happens if the initial condition $q^0 \notin \mathcal{E}_\alpha$?
- The study of the kernel $\mathcal{E}_\alpha$ is not sufficient.

Hodge decomposition and projection on $\mathcal{E}_\alpha$

How can we split a state $q \notin \mathcal{E}_\alpha$?

Theorem

Assume that $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ with $\alpha_{\min} > 0$. We build a Hodge decomposition on the weighted spaces

$$\mathcal{E}_\alpha \oplus \mathcal{E}_\alpha^\perp = L^2_\alpha\left(\mathbb{T}\right)^3,$$

*where the **acoustic space** $\mathcal{E}_\alpha^\perp$ is given by*

$$\mathcal{E}_\alpha^\perp = \left\{ q = (r, \boldsymbol{u})^T \in L^2_\alpha\left(\mathbb{T}\right)^3 \,\middle|\, \int_{\mathbb{T}} r\alpha dx = 0 \text{ and } \exists \phi \in H^1_\alpha\left(\mathbb{T}\right), \boldsymbol{u} = \nabla\phi \right\}$$

Definition

*The Hodge decomposition allows to define an **orthogonale projection***

$$\begin{aligned} \mathbb{P}_\alpha :\ L^2_\alpha\left(\mathbb{T}\right)^3 &\longrightarrow \mathcal{E}_\alpha \\ q &\longmapsto \mathbb{P}_\alpha q. \end{aligned}$$

Structure of the solution of the linear wave equation

Linear wave equation with porosity

$$\begin{cases} \partial_t(\alpha r) + \frac{a_\star}{M} \nabla \cdot (\alpha \mathbf{u}) = 0, \\ \partial_t(\alpha \mathbf{u}) + \frac{a_\star}{M} \alpha \nabla r = 0. \end{cases}$$

Proposition

If q is a solution of the linear wave equation with porosity with an initial condition q^0, we have

$$\forall q^0 \in \mathcal{E}_\alpha, \quad q(t \geq 0) = q^0 \in \mathcal{E}_\alpha \quad et \quad \forall q^0 \in \mathcal{E}_\alpha^\perp, \quad q(t \geq 0) \in \mathcal{E}_\alpha^\perp.$$

Corollary

The solution q of the linear wave equation with porosity with an initial condition q^0 can be written as

$$q = \mathbb{P}_\alpha q^0 + \left(q - \mathbb{P}_\alpha q^0\right) \in \mathcal{E}_\alpha + \mathcal{E}_\alpha^\perp.$$

Structure of the solution of the linear wave equation

Linear wave equation with porosity

$$\begin{cases} \partial_t(\alpha r) + \frac{a_\star}{M}\nabla\cdot(\alpha \mathbf{u}) = 0, \\ \partial_t(\alpha \mathbf{u}) + \frac{a_\star}{M}\alpha\nabla r = 0. \end{cases}$$

Proposition

The energy of the solution q of the linear wave equation with porosity satisfies

$$\frac{d}{dt}\|q\|^2_{L^2_\alpha} = 0.$$

Corollary

The solution q of the linear wave equation with porosity and with an initial condition q^0 satisfies

$$\forall C > 0,\ \|q^0 - \mathbb{P}_\alpha q^0\|_{L^2_\alpha} \leq CM \Rightarrow \forall t \geq 0,\ \|q - \mathbb{P}_\alpha q^0\|_{L^2_\alpha}(t) \leq CM.$$

Accurate schemes at low Mach number

- We transcribe this property at the discrete level for a short time.
- We build Hodge decompositions on triangular and cartesian meshes.
- We obtain discrete orthogonal projections $\mathbb{P}_\alpha^h$ on $\mathcal{E}_\alpha^\square$ and $\mathcal{E}_\alpha^\triangle$.

Definition

*A scheme is **accurate at low Mach number** if the solution q_h given by the scheme satisfies*

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q_h^0 - \mathbb{P}_\alpha^h q_h^0\|_{l^2_\alpha} = C_1 M$$
$$\Rightarrow \forall t \in [0; C_2 M],\ \|q_h - \mathbb{P}_\alpha^h q_h^0\|_{l^2_\alpha}(t) \leq C_3 M,$$

where C_3 does not depend on M.

Initial condition :

- $a_\star = 1.$
- $M = 10^{-4}.$
- $q_h^0 = q_{h,1}^0 + \mathbf{M} q_{h,2}^0$ with

$$\begin{cases} r_{h,1}^0(x,y) = 1, \\ (\alpha \mathbf{u}_1)_h^0 = \nabla_h \times \psi_h, \end{cases} \Rightarrow q_{h,1}^0 \in \mathcal{E}_\alpha^h$$

and

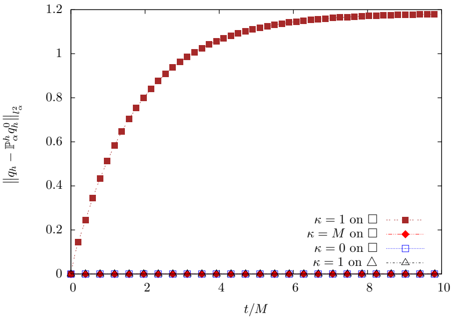
$$\begin{cases} r_{h,2}^0(x,y) = 0, \\ \mathbf{u}_{h,2}^0 = \nabla_h \phi_h, \\ \|q_{h,2}^0\|_{l_\alpha^2} = 1, \end{cases} \Rightarrow q_{h,2}^0 \in \left(\mathcal{E}_\alpha^h\right)^\perp$$

then

$$\|q_h^0 - \mathbb{P}_\alpha q_h^0\|_{l_\alpha^2} = \|M q_{h,2}^0\|_{l_\alpha^2} = M = O(M).$$

- We plot $\|q_h - \mathbb{P}_\alpha q_h^0\|_{l_\alpha^2}(t)$ as a fonction of the time.

Cartesian mesh with $\Delta x = \Delta y = 0.02$ and $M = 0.0001 \ll \Delta x$:



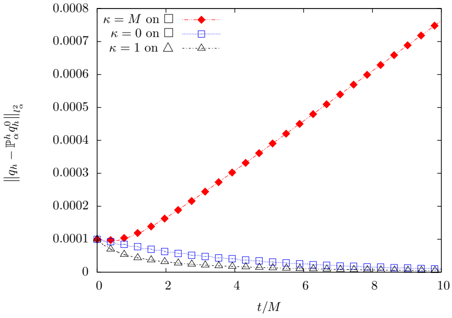
Theorem ($\kappa = 1$ on $\square$)

$$\forall C_1 > 0,\ \exists C_2(C_1) > 0,\ \exists C_3(C_1) > 0,\ \|q^0_h - \mathbb{P}^{h,\square}_\alpha q^0_h\|_{l^2_\alpha} = C_1 M$$

$$\Rightarrow \forall t \geq C_2 M,\ \|q_h - \mathbb{P}^{h,\square}_\alpha q^0_h\|_{l^2_\alpha}(t) \geq C_3 \min(\Delta x, \Delta y),$$

for all $M \leq \frac{C_3}{C_1} \min(\Delta x, \Delta y)$.

Cartesian mesh with $\Delta x = \Delta y = 0.02$ and $M = 0.0001 \ll \Delta x$:

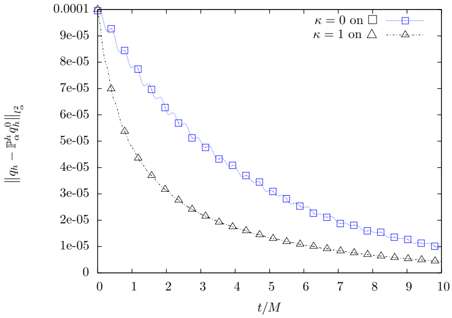


Theorem ($\kappa = M$ on □)

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q_h^0 - \mathbb{P}_\alpha^{h,\square} q_h^0\|_{l^2_\alpha} = C_1 M$$
$$\Rightarrow \forall t \in [0; C_2 M],\ \|q_h - \mathbb{P}_\alpha^{h,\square} q_h^0\|_{l^2_\alpha}(t) \leq C_3 M,$$

where C_3 does not depend on M.

Cartesian mesh with $\Delta x = \Delta y = 0.02$ and $M = 0.0001 \ll \Delta x$:

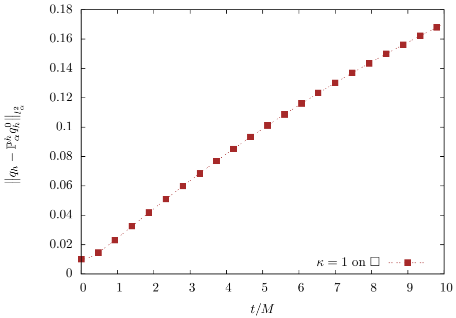


Theorem ($\kappa = 1$ on $\triangle$ et $\kappa = 0$ sur $\square$)

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q^0_h - \mathbb{P}^h_\alpha q^0_h\|_{l^2_\alpha} = C_1 M$$
$$\Rightarrow \forall t \geq 0,\ \|q_h - \mathbb{P}^h_\alpha q^0_h\|_{l^2_\alpha}(t) \leq C_3 M,$$

where C_3 does not depend on M.

Cartesian mesh on $\Delta x = \Delta y = 0.0033$ and $M = 0.01 \gg \Delta x$:



Theorem ($\kappa = 1$ on □)

$$\forall C_0, C_1, C_2 > 0, \exists C_3(C_0, C_1, C_2) > 0, \begin{cases} \Delta x \leq C_0 M, \ et \ \Delta y \leq C_0 M, \\ \left\| q^0_h - \mathbb{P}^h_\alpha q^0_h \right\|_{l^2_\alpha} = C_1 M \end{cases}$$

$$\Rightarrow \forall t \in [0; C_2 M], \left\| q_h - \mathbb{P}^h_\alpha q^0_h \right\|_{l^2_\alpha}(t) \leq C_3 M,$$

where C_3 does not depend on M, Δx and Δy.

Conclusion of the linear case

Triangular mesh :

- The Godunov scheme ($\kappa = 1$) is accurate at low Mach.

Cartesian mesh :

- The Godunov scheme ($\kappa = 1$)
  - is not accurate at low Mach number if $M \ll \min(\Delta x, \Delta y)$.
  - is accurate at low Mach number if $M \gg \min(\Delta x, \Delta y)$.
- Two corrections for low Mach flows :

$$F^{Cor}(W_i, W_j) = F^{God}(W_i, W_j) - \frac{(1-\kappa)a_\star \alpha_i}{2M\alpha_{ij}} \begin{pmatrix} 0 \\ \left[\left((\alpha \mathbf{u})_i - (\alpha \mathbf{u})_j\right) \cdot \mathbf{n}_{ij}\right] \mathbf{n}_{ij} \end{pmatrix}$$

  - $\kappa = 0$ (**low Mach** correction) : accurate at low Mach,
  - $\kappa = \min(M, 1)$ (**all Mach** correction) : accurate at low Mach and allows to obtain the Godunov scheme for $M \geq 1$.

Next step :

- Test the different schemes in the non-linear case.

Extension to the non-linear case with $\alpha = 1$

Euler equations :

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \\ \partial_t (\rho E) + \nabla \cdot ((\rho E + p)\mathbf{u}) = 0 \end{cases}$$

Note $W = (\rho, \rho \mathbf{u}, \rho E)^T$. The numerical scheme can be written as

$$\frac{d}{dt} W_i + \frac{1}{|\Omega_i|} \sum_{\Gamma_{ij} \subset \partial \Omega_i} |\Gamma_{ij}| F(W_i, W_j, \mathbf{n}_{ij}) = 0.$$

The **low Mach** and the **all Mach** corrections consist to replace the flux $F(W_i, W_j, \mathbf{n}_{ij})$ with

$$F^{Cor}(W_i, W_j) = F^{Godunov}(W_i, W_j) - \frac{(1 - \kappa_{ij}) \rho_{ij} c_{ij}}{2} \begin{pmatrix} 0 \\ \left[(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij} \right] \mathbf{n}_{ij} \\ 0 \end{pmatrix}$$

where respectively $\kappa_{ij} = 0$ or $\kappa_{ij} = \min \left(1, \frac{|u_{ij}|}{c_{ij}} \right)$.

2D low Mach flow : vortex in a box ($\alpha = 1$)

Tools :

- *Mesh :* the software Salome,
- *Code :* Librairy C++ CDMATH (http://www.cdmath.jimdo.com).

Initial condition :

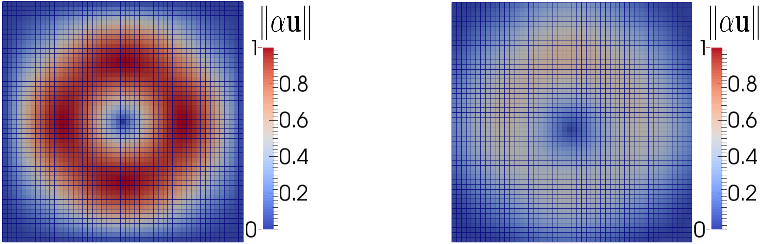
- We test the accuracy of the **low Mach** and the **all Mach** scheme for a low Mach flow.
- The initial state is given on the domain $[0,1] \times [0,1]$ by

$$\begin{cases} \rho = 1, \\ \mathbf{u} = \nabla \times \psi \quad \text{where} \quad \psi(x,y) = \frac{1}{\pi}\sin^2(\pi x)\sin^2(\pi y), \\ p = 1000. \end{cases}$$

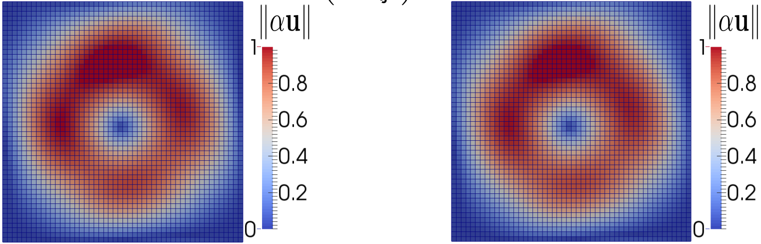
- Wall boundary conditions.
- Final time of computation of $t_{final} = 0.125s$.
- *Mach* ≈ 0.026.

2D low Mach flow : vortex in a box ($\alpha = 1$)

Initial condition and Godunov ($\kappa_{ij} = 1$) on a 50×50 cartesian grid :

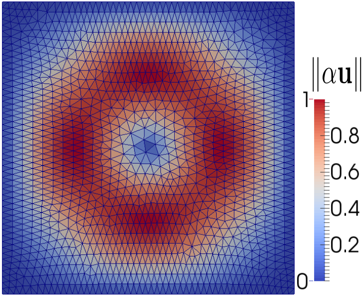


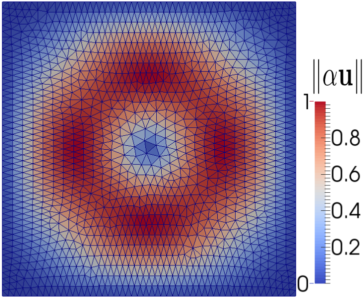
$\kappa_{ij} = 0$ (LM) and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) on a cartesian 50×50 :



2D low Mach flow : vortex in a box ($\alpha = 1$)

Initial condition and Godunov with $\kappa_{ij} = 1$ on a triangular mesh :





- The **low Mach** and the **all Mach** (AM) schemes are accurate at low Mach number.

2D compressible flow : 2D Riemann problem ($\alpha = 1$)

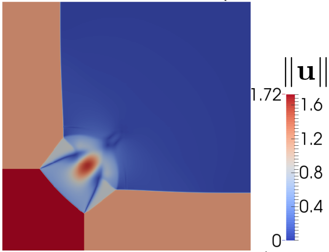
- We test the stability of the **low Mach** and **all Mach** scheme for a compressible flow ($0 \leq Mach \leq 3.14.$).
- The initial state is given on the domain $[0,1] \times [0,1]$ by

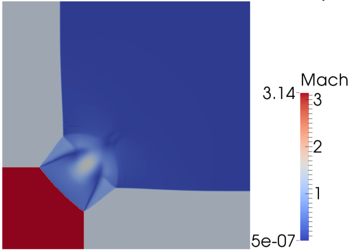
$$(\rho, u_x, u_y)(x,y) = \begin{cases} (0.1380, 1.206, 1.206), & \text{for } x < 0.5, \quad y < 0.5 \\ (0.5323, 0.000.1.206), & \text{for } x > 0.5, \quad y < 0.5 \\ (0.5323, 1.206, 0.000), & \text{for } x < 0.5, \quad y > 0.5 \\ (1.5000, 0.000, 0.000), & \text{for } x > 0.5, \quad y > 0.5. \end{cases}$$

  (4 shock wave interaction).
- Exact boundary conditions.
- Final time of computation : $t_{final} = 0.4s$.

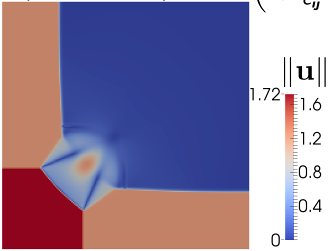
2D compressible flow : 2D Riemann problem ($\alpha = 1$)

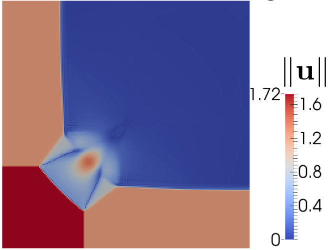
Reference solution (Gudunov ($\kappa_{ij} = 1$) on a 600×600 cartesian grid) :





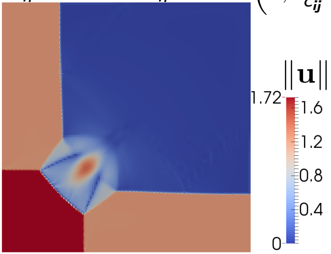
$\kappa_{ij} = 1$ and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) on a 200×200 cartesian grid :

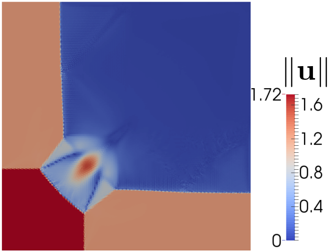




2D compressible flow : 2D Riemann problem ($\alpha = 1$)

$\kappa_{ij} = 1$ and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) on a 40 000 triangular mesh :

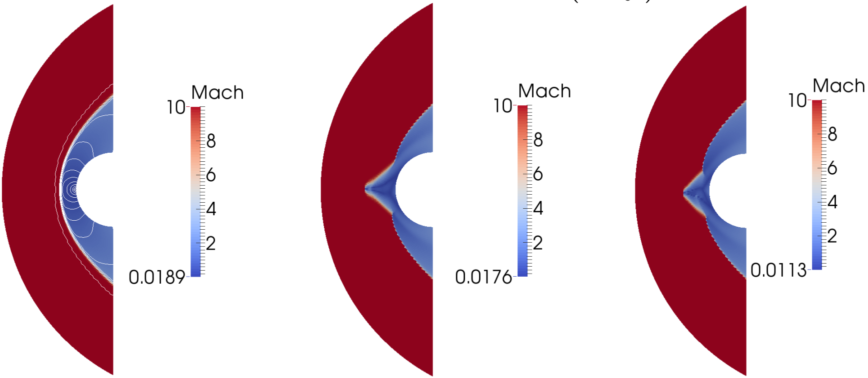




- The **all Mach** scheme is stable on triangular and cartesian meshes for this compressible flow.
- The **low Mach** scheme is not stable for this compressible flow.

Carbuncle phenomena : supersonic flow around a cylinder

- We initialize with a $Mach = 10$ flow on a 80×160 radial mesh.
- Physical solution, $\kappa_{ij} = 1$ and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) :



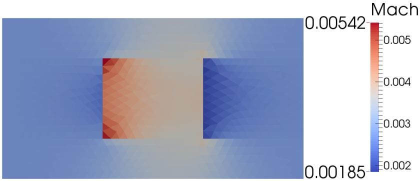
- The **all Mach** scheme is stable for a supersonic flow around a cylinder but also produces the carbuncle phenomena...

A first test with $\alpha \neq 1$: a low Mach flow

- We use a **VFRoe solver** in variables $(\alpha, \rho, \alpha\mathbf{u}, p)$.
- We test the accuracy of the **all Mach** scheme for a low Mach flow with porosity $(0 \leq Mach \leq 5 \times 10^{-3}.)$.
- The initial state is given on the domain $[0, 1.5] \times [0, 8]$ by

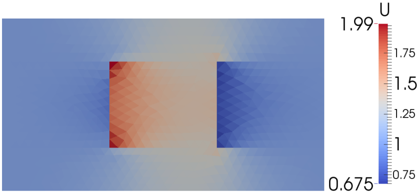
$$\rho = 1, \quad u_x = 1, \quad u_y = 0, \quad p = 10^5,$$
$$\alpha = \begin{cases} 0.5, & \text{for } (x, y) \in [0.5; 1] \times [0.2; 0.6], \\ 1, & \text{otherwise.} \end{cases}$$

- Final time of computation : $t_{final} = 0.05s$.

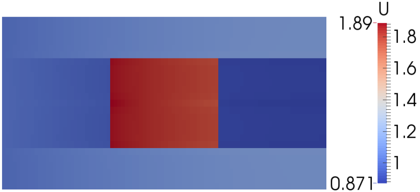


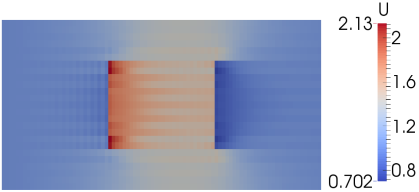
A first test with $\alpha \neq 1$: a low Mach flow

$\kappa_{ij} = 1$ on a triangular mesh (reference solution) :



$\kappa_{ij} = 1$ and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) on a cartesian mesh :





Final conclusion and perspective

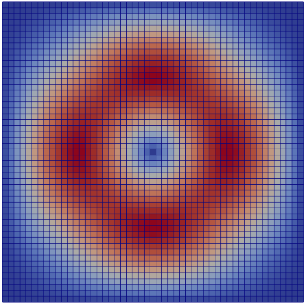
Conclusion :

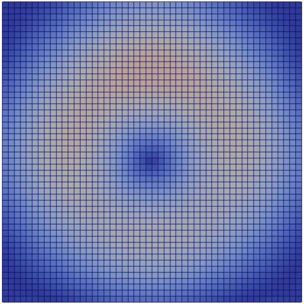
- We note the accuracy at low Mach number of finite volume schemes on triangular meshes in the linear and non-linear cases.
- We note the inaccuracy at low Mach number of the finite volume schemes on cartesian meshes in the linear and non-linear cases.
- The study of the linear case allows to propose a correction that gives an accurate scheme at low Mach number and gives the Godunov scheme when $M \geq 1$ on cartesian meshes.

Perpectives :

- Test the scheme with non-constant fonction α in the non-linear case.
- Study the stability of the corrected scheme.

Thank you for your attention !





S. Dellacherie, P. Omnes. On the Godunov scheme applied to the variable cross-section linear equation. FVCA6, (4) :313–321, 2011.