Construction of modified Godunov type schemes accurate at any Mach number for the compressible Euler system

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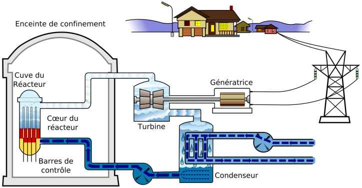
Study case :

- Nuclear core reactor.



Properties of the flow :

- Low Mach flow :

$$\begin{cases} |\mathbf{u}| & \approx & 5\ m.s^{-1}, \\ c & \approx & 500\ m.s^{-1}, \end{cases}$$

$$\Rightarrow M := \frac{|\mathbf{u}|}{c} \approx 10^{-2} \ll 1.$$

- Compressible flow : shock wave in some accidental cases.

Aim :

- Develop a "compressible" numerical scheme that is accurate at low Mach number.

Barotropic Euler equations and incompressible states

- Compressible barotropic Euler equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \end{cases}$$

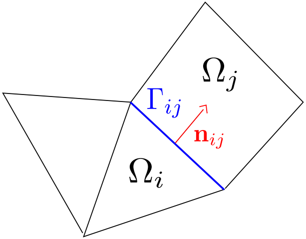
where $p = p(\rho)$ and $p'(\rho) > 0$.

- Conservative forme : $\partial_t W + \nabla \cdot (\mathbf{F}(W)) = 0$.
- Hyperbolic system under the condition $p'(\rho) > 0$ $\left(c = \sqrt{p'(\rho)}\right)$.
- Study the behavior of our scheme for incompressible states

$$\mathcal{E} := \left\{ W \in \mathbb{R}^2 | \rho = cst \text{ and } \nabla \cdot \mathbf{u} = 0 \right\}.$$

Godunov scheme (Finite volume scheme)

We integrate $\partial_t W + \nabla \cdot (\mathbf{F}(W)) = 0$ on each cell Ω_i :



$$\frac{d}{dt}\int_{\Omega_i} W\, d\mathbf{x} + \int_{\Omega_i} \nabla \cdot (\mathbf{F}(W))\, d\mathbf{x} = 0$$

$$\Rightarrow \frac{d}{dt} W_i + \sum_{\Gamma_{ij} \subset \partial\Omega_i} \int_{\Gamma_{ij}} \mathbf{F}(W) \cdot \mathbf{n}_{ij}\, d\sigma = 0$$

$$\Rightarrow \frac{d}{dt} W_i + \sum_{\Gamma_{ij} \subset \partial\Omega_i} |\Gamma_{ij}| \mathbf{F}(W_{ij}) \cdot \mathbf{n}_{ij}\, d\sigma = 0$$

where W_{ij} is the solution of the 1D Riemann problem

$$\begin{cases} \partial_t W + \partial_\xi F(W) = 0, \\ W(t=0,\xi) = \begin{cases} W_i, & \text{if } \xi < 0, \\ W_j, & \text{otherwise.} \end{cases} \end{cases}$$

on $\xi/t = 0$ where ξ is the coordinate in the $\mathbf{n}_{ij}$ direction.

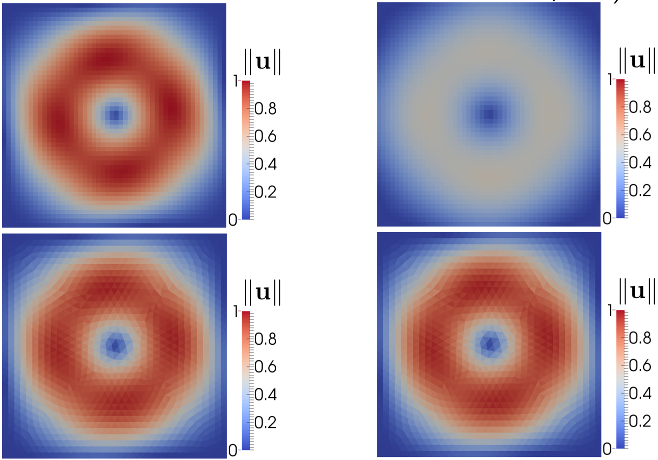
The low Mach numerical problem

Results obtained with the Godunov scheme for an initial incompressible state $W^0 \in \mathcal{E}$ ($\rho^0 = cst$ and $\nabla \cdot \mathbf{u}^0 = 0$) :

Initial condition : | At $t = 0.0125(\approx M)$:

Cartesian mesh

Triangular mesh



Outline

Aims :

- Explain the different behavior on cartesian and triangular meshes.
- Propose a correction on cartesian meshes.

Outline :

- Linearize the system to simplify the study.
- Understand the numerical problem at low Mach number on the linearized system.
- Propose a corrected scheme that is accurate at low Mach number on cartesian meshes for the linearized system.
- Extend the correction to the non-linear case and test it.

Linear wave equation

- Dimensionless : we introduce $\tilde{x} = \frac{x}{L}$, $\tilde{y} = \frac{y}{L}$, $\tilde{t} = \frac{t}{T}$, $\tilde{\rho} = \frac{\rho}{\rho_0}$, $\tilde{u_x} = \frac{u_x}{u_0}$, $\tilde{u_y} = \frac{u_y}{u_0}$, $\tilde{p} = \frac{p}{p_0}$ avec $u_0 = \frac{L}{T}$, we obtain

$$\begin{cases} \partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}}) = 0, \\ \partial_{\tilde{t}}(\tilde{\rho}\tilde{\mathbf{u}}) + \tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}) + \dfrac{1}{M^2}\nabla\tilde{p} = 0, \end{cases} \quad \text{with } M = \frac{u_0}{c_0}.$$

- Change of variable $\tilde{\rho} := \tilde{\rho}_\star\left(1 + \frac{M}{a_\star}\tilde{r}\right)$, with $\begin{cases} a_\star^2 = \tilde{p}'(\tilde{\rho}_\star) \\ \frac{M}{a_\star}\tilde{r} \ll 1. \end{cases}$

$$\begin{cases} \partial_t\tilde{r} + \tilde{\nabla}\cdot(\tilde{r}\tilde{\mathbf{u}}) + \frac{a_\star}{M}\tilde{\nabla}\cdot\tilde{\mathbf{u}} = 0, \\ \partial_{\tilde{t}}(\tilde{\mathbf{u}}) + (\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{\mathbf{u}} + \frac{1}{M}\dfrac{\tilde{p}'\left(\tilde{\rho}_\star\left(1+\frac{M}{a_\star}\tilde{r}\right)\right)}{a_\star\left(1+\frac{M}{a_\star}\tilde{r}\right)}\nabla\tilde{r} = 0. \end{cases}$$

- Linearization around $(\tilde{r} = 0, \tilde{\mathbf{u}} = 0)$: **linear wave equation**

$$\begin{cases} \partial_t r + \frac{a_\star}{M}\nabla\cdot\mathbf{u} = 0, \\ \partial_t\mathbf{u} + \frac{a_\star}{M}\nabla r = 0. \end{cases}$$

- **Linear wave equation**

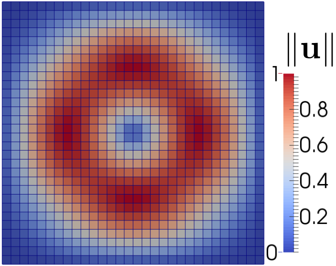
$$\begin{cases} \partial_t r + \frac{a_\star}{M}\nabla\cdot\mathbf{u} = 0, \\ \partial_t \mathbf{u} + \frac{a_\star}{M}\nabla r = 0. \end{cases}$$

- Kernel of the spatial operator : **incompressible space**

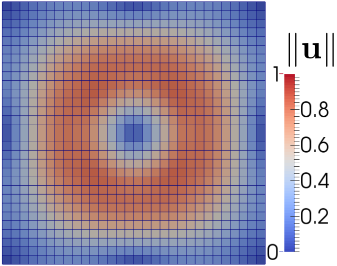
$$\mathcal{E} := \left\{ q = (r, \mathbf{u})^T \in L^2(\mathbb{T})^3 \,\middle|\, \nabla r = 0 \text{ and } \nabla\cdot\mathbf{u} = 0 \right\}.$$

- Same behavior with the linear Godunov scheme on cartesian meshes :

Initial condition ($q^0 \in \mathcal{E}$) :



At $t = 0.0001(= M)$:



Explanation of the wrong behavior on cartesian meshes

Godunov scheme on any mesh $(\Omega_i)_{1\leq i\leq N}$:

$$\begin{cases} \dfrac{d}{dt} r_i + \dfrac{a_\star}{2M}\dfrac{1}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}|\Big[(\mathbf{u}_i+\mathbf{u}_j)\cdot\mathbf{n}_{ij} + (r_i - r_j)\Big] = 0, \\ \dfrac{d}{dt}\mathbf{u}_i + \dfrac{a_\star}{2M}\dfrac{1}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}|\Big[r_i + r_j + \kappa(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}\Big]\mathbf{n}_{ij} = 0 \end{cases} \quad \text{with } \kappa = 1.$$

Modified equation on a cartesian mesh :

- The Godunov scheme on a cartesian mesh $\Omega_{i,j}$ for r gives us

$$\partial_t r_{i,j} + \frac{a_\star}{M}\frac{(u_x)_{i+1,j}-(u_x)_{i-1,j}}{2\Delta x} + \frac{a_\star}{M}\frac{(u_y)_{i,j+1}-(u_y)_{i,j-1}}{2\Delta y}$$
$$= \frac{a_\star}{2M\Delta x}\left(r_{i+1,j} - 2r_{i,j} + r_{i-1,j}\right) + \frac{a_\star}{2M\Delta y}\left(r_{i,j+1} - 2r_{i,j} + r_{i,j-1}\right).$$

- Then, the first order modified equation for r is

$$\partial_t r + \frac{a_\star}{M}\nabla\cdot\mathbf{u} = \frac{a_\star\Delta x}{2M}\partial_x^2 r + \frac{a_\star\Delta y}{2M}\partial_y^2 r.$$

Explanation of the wrong behavior on cartesian meshes

Modified equation on a cartesian mesh :

- We use the same method for **u** and we obtain the modified system

$$\partial_t r + \frac{a_\star}{M}\nabla \cdot \mathbf{u} = \frac{a_\star \Delta x}{2M}\partial_x^2 r + \frac{a_\star \Delta y}{2M}\partial_y^2 r$$

$$\partial_t \mathbf{u} + \frac{a_\star}{M}\nabla r = \begin{pmatrix} \kappa \dfrac{a_\star \Delta x}{2M}\partial_x^2 u_x \\ \kappa \dfrac{a_\star \Delta y}{2M}\partial_y^2 u_y \end{pmatrix}$$

with $\kappa = 1$. We write it as

$$\partial_t q + \frac{\mathcal{L}_\kappa}{M}(q) = 0 \quad \text{with} \quad \mathcal{L}_\kappa = L_\alpha - MB_\kappa.$$

- **What is the relation between Ker $\mathcal{L}_\kappa$ and $\mathcal{E}$?**

Explanation of the wrong behavior on cartesian meshes

Proposition

1. *If* $\kappa > 0$, *we have*

$$Ker\,\mathcal{L}_{\kappa>0} = \left\{q := (r, \boldsymbol{u})^T | \nabla r = 0 \text{ and } \partial_x u_x = \partial_y u_y = 0\right\} \subsetneq \mathcal{E}.$$

2. *If* $\kappa = 0$, *we have* $Ker\,\mathcal{L}_{\kappa=0} = \mathcal{E}$.

Conclusion of the study of the continuous case :

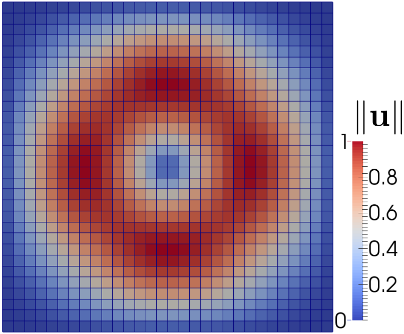
- Substitute $\kappa = 1$ by $\kappa = 0$ seams to allow to the Godunov scheme to preserve the incompressible states $q^0 \in \mathcal{E}$ on cartesian meshes.

To do :

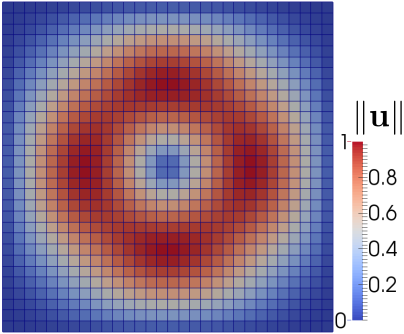
- Test this correction ($\kappa = 0$) at the discret level.

Test of the low Mach correction $\kappa = 0$

- Initial condition $q^0 \in \mathcal{E}$:



- At $t = 0.0001 (= M)$:



Next step :

- Study the problem at the discrete level.
- Study the case of a triangular mesh.

Discrete study

- We define the spaces $\mathcal{E}^{\triangle}$ and $\mathcal{E}^{\square}$ associated to the incompressible space $\mathcal{E}$ on a triangular and a cartesian mesh.
- Recall the Godunov scheme on a triangular or cartesian mesh $(\Omega_i)_{1\leq i\leq N}$

$$\begin{cases} \dfrac{d}{dt} r_i + \dfrac{a_\star}{2M} \dfrac{1}{|\Omega_i|} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| \Big[(\mathbf{u}_i + \mathbf{u}_j) \cdot \mathbf{n}_{ij} + (r_i - r_j) \Big] = 0, \\ \dfrac{d}{dt} \mathbf{u}_i + \dfrac{a_\star}{2M} \dfrac{1}{|\Omega_i|} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| \Big[r_i + r_j + \kappa (\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij} \Big] \mathbf{n}_{ij} = 0 \end{cases}$$

with $\kappa = 1$.

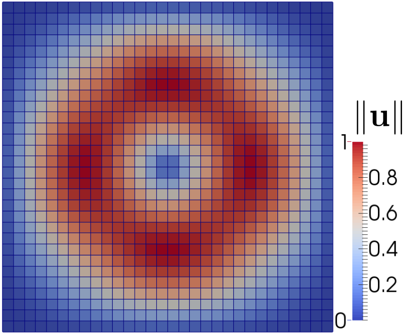
- We write it as

$$\frac{d}{dt} q_h + \frac{\mathbb{L}_\kappa^h}{M}(q_h) = 0,$$

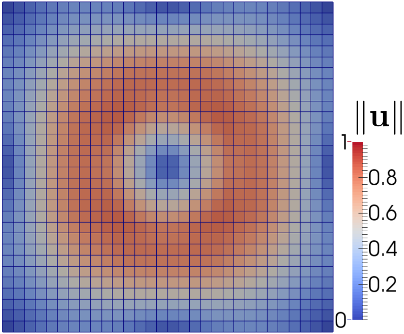
where $q_h = (r_i, \mathbf{u}_i)^T$.

Gudunov scheme $(\kappa = 1)$ on a cartesian mesh :

- Initial condition $q^0 \in \mathcal{E}_\alpha^\square$:



- At $t = 0.0001 (= M)$:

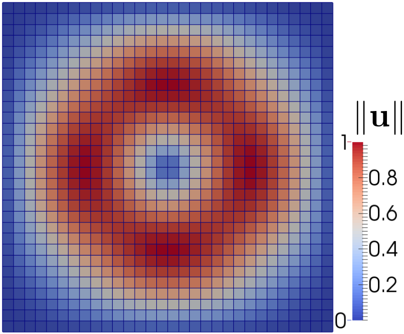


Proposition $(\kappa = 1$ on $\square)$

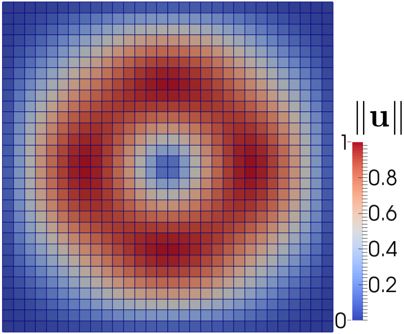
$$Ker\,\mathbb{L}_{\kappa=1}^{h,\square} \subsetneq \mathcal{E}^\square.$$

Low Mach scheme $(\kappa = 0)$ on a cartesian mesh :

- Initial condition $q^0 \in \mathcal{E}^{\square}$:



- At $t = 0.0001 (= M)$:

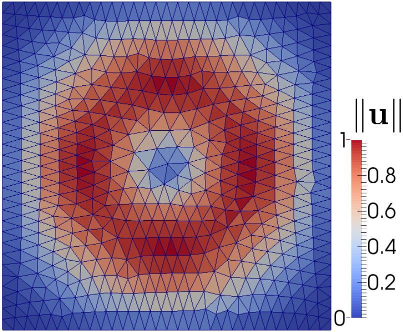


Proposition $(\kappa = 0$ on $\square)$

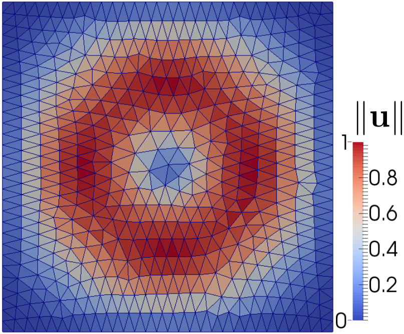
$$Ker\, \mathbb{L}^{h,\square}_{\kappa=0} = \mathcal{E}^{\square}.$$

Godunov scheme $(\kappa = 1)$ on a triangular mesh :

- Initial condition $q^0 \in \mathcal{E}^{\triangle}$:



- At $t = 0.0001 (= M)$:



Proposition $(\kappa = 1$ on $\triangle)$

$$Ker\, \mathbb{L}^{h,\triangle}_{\kappa=1} = \mathcal{E}^{\triangle}.$$

Conclusion of the discrete study of the kernel of the Godunov scheme

Conclusion :

- The Godunov scheme ($\kappa = 1$) does not preserve some incompressible states $\mathcal{E}^{\square}$ on a cartesian mesh.
- The low Mach scheme ($\kappa = 0$) preserves the incompressible states $\mathcal{E}^{\square}$ on a cartesian mesh.
- The Godunov scheme ($\kappa = 1$) preserves the incompressible states $\mathcal{E}^{\triangle}$ on a triangular mesh.

BUT :

- We wish a correction that allows to obtain the Godunov scheme when the Mach number tends to 1.
- What happens if the initial condition $q^0 \notin \mathcal{E}$?
- The study of the kernel $\mathcal{E}$ is not sufficient.

Hodge decomposition and projection on $\mathcal{E}$

How can we split a state $q \notin \mathcal{E}$?

Theorem

We use the Hodge decomposition

$$\mathcal{E}_\alpha \oplus \mathcal{E}^\perp = L^2(\mathbb{T})^3,$$

where the ***acoustic space*** *$\mathcal{E}^\perp$ is given by*

$$\mathcal{E}^\perp = \left\{ q = (r, \boldsymbol{u})^T \in L^2(\mathbb{T})^3 \,\middle|\, \int_{\mathbb{T}} r dx = 0 \text{ and } \exists \phi \in H^1(\mathbb{T}), \boldsymbol{u} = \nabla\phi \right\}$$

Definition

The Hodge decomposition allows to define an ***orthogonale projection***

$$\begin{aligned} \mathbb{P} : \; L^2(\mathbb{T})^3 &\longrightarrow \mathcal{E} \\ q &\longmapsto \mathbb{P}q. \end{aligned}$$

Structure of the solution of the linear wave equation

Linear wave equation

$$\begin{cases} \partial_t r + \frac{a_\star}{M} \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \frac{a_\star}{M} \nabla r = 0. \end{cases}$$

Proposition

If q is a solution of the linear wave equation with an initial condition q^0, we have

$$\forall q^0 \in \mathcal{E}, \quad q(t \geq 0) = q^0 \in \mathcal{E} \quad \text{and} \quad \forall q^0 \in \mathcal{E}^\perp, \quad q(t \geq 0) \in \mathcal{E}^\perp.$$

Corollary

The solution q of the linear wave equation with an initial condition q^0 can be written as

$$q = \mathbb{P} q^0 + \left(q - \mathbb{P} q^0\right) \in \mathcal{E} + \mathcal{E}^\perp.$$

Structure of the solution of the linear wave equation

Linear wave equation with porosity

$$\begin{cases} \partial_t r + \frac{a_\star}{M} \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \frac{a_\star}{M} \nabla r = 0. \end{cases}$$

Proposition

The energy of the solution q of the linear wave equation satisfies

$$\frac{d}{dt} \|q\|_{L^2}^2 = 0.$$

Corollary

The solution q of the linear wave equation and with an initial condition q^0 satisfies

$$\forall C > 0, \ \|q^0 - \mathbb{P} q^0\|_{L^2} \leq CM \Rightarrow \forall t \geq 0, \ \|q - \mathbb{P} q^0\|_{L^2}(t) \leq CM.$$

Accurate schemes at low Mach number

- We transcribe this property at the discrete level for short time.
- We build Hodge decompositions on triangular and cartesian meshes.
- We obtain discrete orthogonal projections $\mathbb{P}^h$ on $\mathcal{E}^{\square}$ and $\mathcal{E}^{\triangle}$.

Definition

A scheme is ***accurate at low Mach number*** *if the solution q_h given by the scheme satisfies*

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q_h^0 - \mathbb{P}^h q_h^0\|_{l^2} = C_1 M$$
$$\Rightarrow \forall t \in [0; C_2 M],\ \|q_h - \mathbb{P}^h q_h^0\|_{l^2}(t) \leq C_3 M,$$

where C_3 depends on q^0, $\mathbb{T}$, C_1 and C_2 but does not depend on M.

Initial condition :

- $a_\star = 1$.
- $M = 10^{-4}$.
- $q_h^0 = q_{h,1}^0 + \mathbf{M} q_{h,2}^0$ with

$$\begin{cases} r_{h,1}^0(x,y) = 1, \\ (\mathbf{u}_1)_h^0 = \nabla_h \times \psi_h, \end{cases} \Rightarrow q_{h,1}^0 \in \mathcal{E}^h$$

and

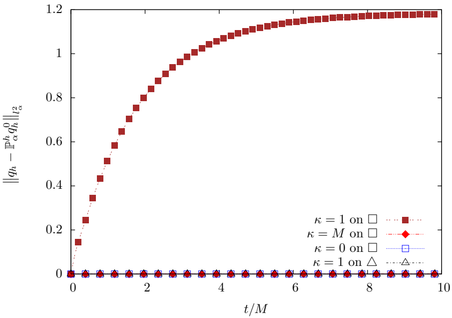
$$\begin{cases} r_{h,2}^0(x,y) = 0, \\ \mathbf{u}_{h,2}^0 = \nabla_h \phi_h, \\ \|q_{h,2}^0\|_{l^2} = 1, \end{cases} \Rightarrow q_{h,2}^0 \in \left(\mathcal{E}^h\right)^\perp$$

then

$$\|q_h^0 - \mathbb{P} q_h^0\|_{l^2} = \|M q_{h,2}^0\|_{l^2} = M = O(M).$$

- We plot $\|q_h - \mathbb{P} q_h^0\|_{l^2}(t)$ as a fonction of the time.

Cartesian mesh with $\Delta x = \Delta y = 0.02$ and $M = 0.0001 \ll \Delta x$:

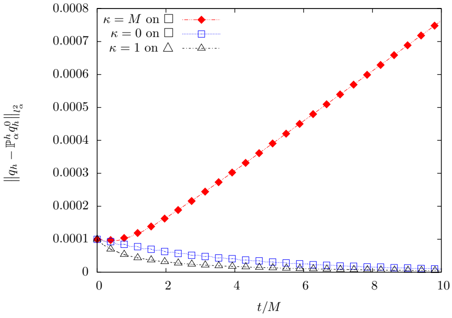


Theorem ($\kappa = 1$ on $\square$)

$$\forall C_1 > 0,\ \exists C_2(C_1) > 0,\ \exists C_3(C_1) > 0,\ \|q_h^0 - \mathbb{P}^{h,\square} q_h^0\|_{l^2} = C_1 M$$
$$\Rightarrow \forall t \geq C_2 M,\ \|q_h - \mathbb{P}^{h,\square} q_h^0\|_{l^2}(t) \geq C_3 \min(\Delta x, \Delta y),$$

for all $M \leq \frac{C_3}{C_1} \min(\Delta x, \Delta y)$.

Cartesian mesh with $\Delta x = \Delta y = 0.02$ and $M = 0.0001 \ll \Delta x$:

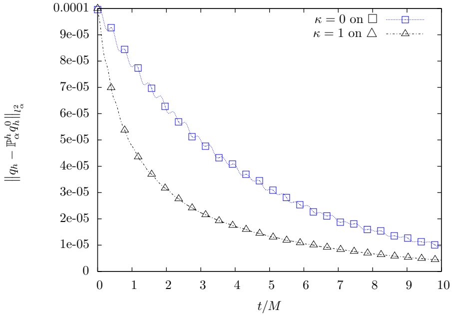


Theorem ($\kappa = M$ on $\square$)

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q_h^0 - \mathbb{P}^{h,\square} q_h^0\|_{l^2} = C_1 M$$
$$\Rightarrow \forall t \in [0; C_2 M],\ \|q_h - \mathbb{P}^{h,\square} q_h^0\|_{l^2}(t) \le C_3 M,$$

where C_3 does not depend on M.

Cartesian mesh with $\Delta x = \Delta y = 0.02$ and $M = 0.0001 \ll \Delta x$:

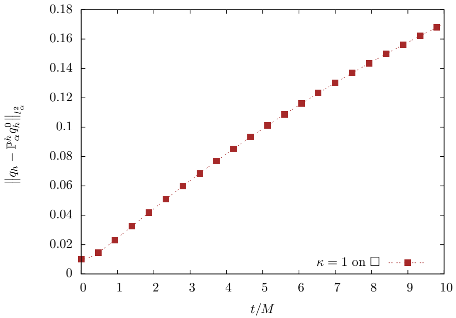


Theorem ($\kappa = 1$ on $\triangle$ et $\kappa = 0$ sur $\square$)

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q_h^0 - \mathbb{P}^h q_h^0\|_{l^2} = C_1 M$$
$$\Rightarrow \forall t \geq 0,\ \|q_h - \mathbb{P}^h q_h^0\|_{l^2}(t) \leq C_3 M,$$

where C_3 does not depend on M.

Cartesian mesh on $\Delta x = \Delta y = 0.0033$ and $M = 0.01 \gg \Delta x$:



Theorem ($\kappa = 1$ on □)

$$\forall C_0, C_1, C_2 > 0, \exists C_3(C_0, C_1, C_2) > 0, \begin{cases} \Delta x \leq C_0 M, \ et \ \ \Delta y \leq C_0 M, \\ \left\| q_h^0 - \mathbb{P}^h q_h^0 \right\|_{l^2} = C_1 M \end{cases}$$

$$\Rightarrow \forall t \in [0; C_2 M], \ \left\| q_h - \mathbb{P}^h q_h^0 \right\|_{l^2}(t) \leq C_3 M,$$

where C_3 does not depend on M, Δx and Δy.

Conclusion of the linear case

Triangular mesh :

- The Godunov scheme ($\kappa = 1$) is accurate at low Mach.

Cartesian mesh :

- The Godunov scheme ($\kappa = 1$)
  - is not accurate at low Mach number if $M \ll \min(\Delta x, \Delta y)$.
  - is accurate at low Mach number if $M \gg \min(\Delta x, \Delta y)$.
- Two corrections for low Mach flows :

$$F^{Cor}(W_i, W_j) = F^{God}(W_i, W_j) - \frac{(1-\kappa)a_\star}{2M} \begin{pmatrix} 0 \\ [(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij}]\, \mathbf{n}_{ij} \end{pmatrix}$$

  - $\kappa = 0$ (**low Mach** correction) : accurate at low Mach,
  - $\kappa = \min(M, 1)$ (**all Mach** correction) : accurate at low Mach and allows to obtain the Godunov scheme for $M \geq 1$.

Next step :

- Add the transport part in the linear wave equation and study the stability of the two corrected schemes.

Some linear stability results

- A linearization of the dimensionless barotropic Euler equation around $(r = 0, \mathbf{u} = \mathbf{u}_\star)$ gives us :

$$\begin{cases} \partial_t r + \mathbf{u}_\star \cdot \nabla r + \frac{a_\star}{M} \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u}_\star \cdot \nabla)\mathbf{u} + \frac{a_\star}{M} \nabla r = 0. \end{cases}$$

- We define the energy as $E_h = \sum_i |\Omega_i| \left[r_i^2 + |u_i|^2 \right]$.

Theorem

Under the subsonic condition $|u_\star| < \frac{a_\star}{M}$, we have :

- *if $\kappa = 1$ (Godunov scheme) : $\frac{d}{dt} E_h \leq 0$.*
- *if $\kappa = \frac{|\boldsymbol{u}_\star|}{a_\star} M$ (all Mach scheme) : $\frac{d}{dt} E_h \leq 0$.*
- *if $\kappa = 0$ (low Mach scheme) :*

$$\frac{d}{dt} E_h \leq \sum_{\Gamma_{ij}} |\boldsymbol{u}_\star \cdot \boldsymbol{n}_{ij}| \left| (\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{n}_{ij} \right|^2 .$$

Extension to the non-linear Euler equations

Euler equations :

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \\ \partial_t (\rho E) + \nabla \cdot ((\rho E + p) \mathbf{u}) = 0 \end{cases}$$

Note $W = (\rho, \rho \mathbf{u}, \rho E)^T$. The numerical scheme can be written as

$$\frac{d}{dt} W_i + \frac{1}{|\Omega_i|} \sum_{\Gamma_{ij} \subset \partial \Omega_i} |\Gamma_{ij}| F(W_i, W_j, \mathbf{n}_{ij}) = 0.$$

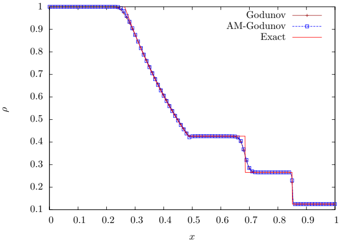
The **low Mach** and the **all Mach** corrections consist to replace the flux $F(W_i, W_j, \mathbf{n}_{ij})$ with

$$F^{Cor}(W_i, W_j) = F^{Godunov}(W_i, W_j) - \frac{(1 - \kappa_{ij}) \rho_{ij} c_{ij}}{2} \begin{pmatrix} 0 \\ [(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij}] \, \mathbf{n}_{ij} \\ 0 \end{pmatrix}$$

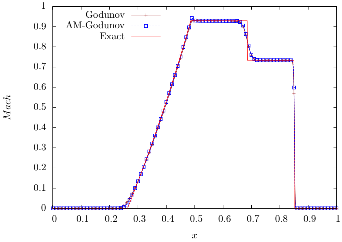
where respectively $\kappa_{ij} = 0$ or $\kappa_{ij} = \min \left(1, \frac{|u_{ij}|}{c_{ij}} \right)$.

1D compressible flow : Sod shock tube

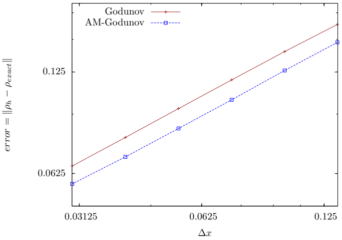
Density ($t = 0.2s$) :



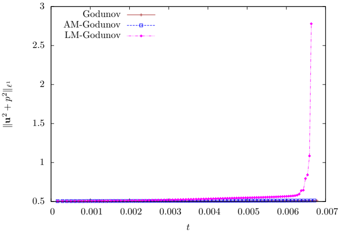
Mach number ($t = 0.2s$) :



Convergence study ($t = 0.2s$) :



Stability :



2D low Mach flow : vortex in a box

Tools :

- *Mesh :* the software Salome,
- *Code :* Librairy C++ CDMATH (http://www.cdmath.jimdo.com).

Initial condition :

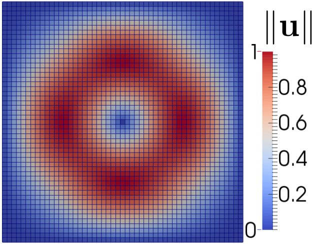
- The initial state is given on the domain $[0,1] \times [0,1]$ by

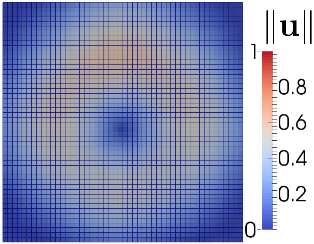
$$\begin{cases} \rho = 1, \\ \mathbf{u} = \nabla \times \psi \quad \text{where} \quad \psi(x,y) = \frac{1}{\pi} \sin^2(\pi x) \sin^2(\pi y), \\ p = 1000. \end{cases}$$

- Wall boundary conditions.
- Final time of computation of $t_{final} = 0.125s$.
- *Mach* ≈ 0.026.

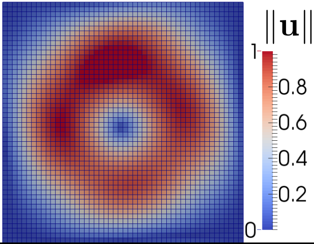
2D low Mach flow : vortex in a box

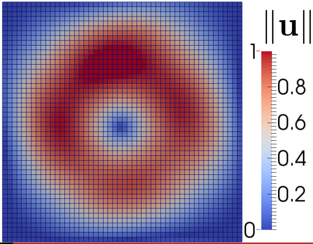
Initial condition and Godunov ($\kappa_{ij} = 1$) on a 50×50 cartesian grid :





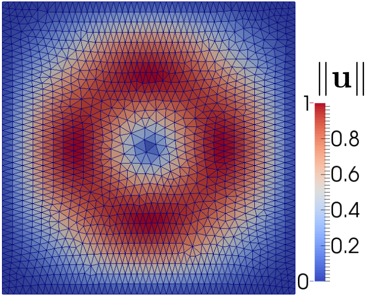
$\kappa_{ij} = 0$ (LM) and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) on a cartesian 50×50 :

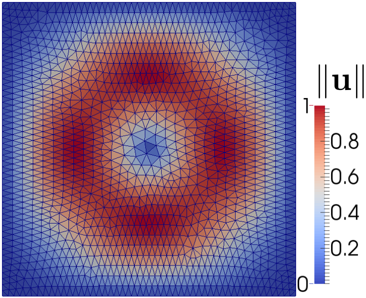




2D low Mach flow : vortex in a box

Initial condition and Godunov with $\kappa_{ij} = 1$ on a triangular mesh :





2D compressible flow : 2D Riemann problem

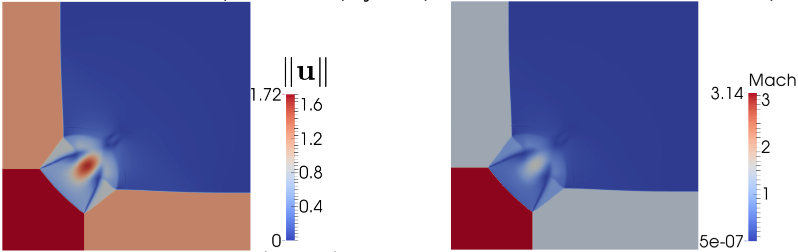
- The initial state is given on the domain $[0,1] \times [0,1]$ by

$$(\rho, u_x, u_y)(x,y) = \begin{cases} (0.1380, 1.206, 1.206), & \text{for } x < 0.5, \quad y < 0.5 \\ (0.5323, 0.000.1.206), & \text{for } x > 0.5, \quad y < 0.5 \\ (0.5323, 1.206, 0.000), & \text{for } x < 0.5, \quad y > 0.5 \\ (1.5000, 0.000, 0.000), & \text{for } x > 0.5, \quad y > 0.5. \end{cases}$$

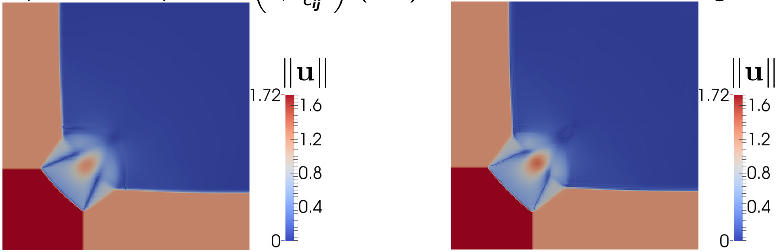
  (4 shock wave interaction).
- Neumann boundary conditions.
- Final time of computation : $t_{final} = 0.4s$.
- $0 \leq Mach \leq 3.14$.

2D compressible flow : 2D Riemann problem

Reference solution (Gudunov ($\kappa_{ij} = 1$) on a 600×600 cartesian grid) :

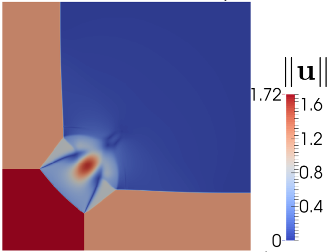


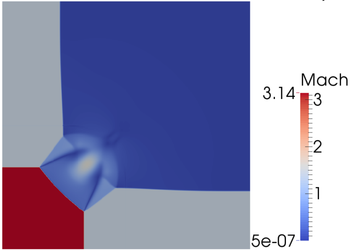
$\kappa_{ij} = 1$ and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) on a 200×200 cartesian grid :



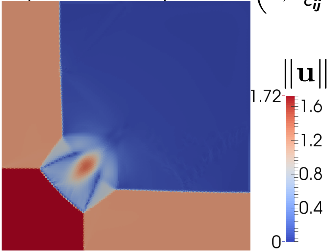
2D compressible flow : 2D Riemann problem

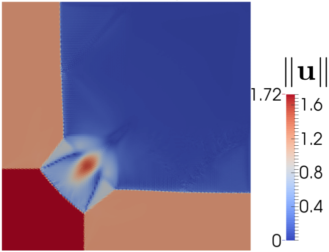
Reference solution (Gudunov ($\kappa_{ij} = 1$) on a 600×600 cartesian grid) :





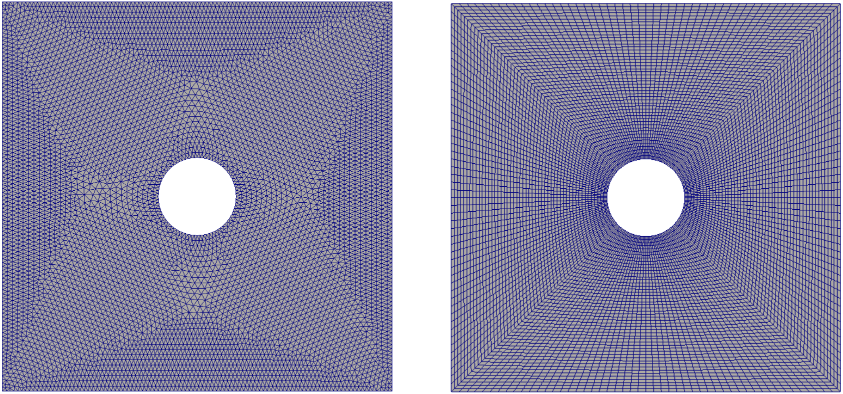
$\kappa_{ij} = 1$ and $\kappa_{ij} = \min\left(1, \frac{|u_{ij}|}{c_{ij}}\right)$ (AM) on a 40 000 triangular mesh :





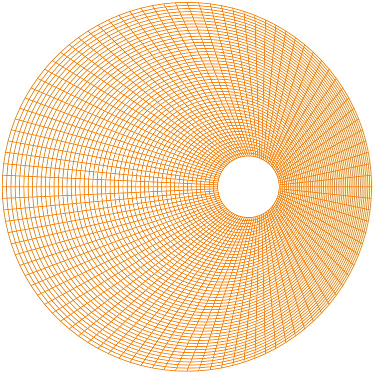
Steady flow around a cylinder

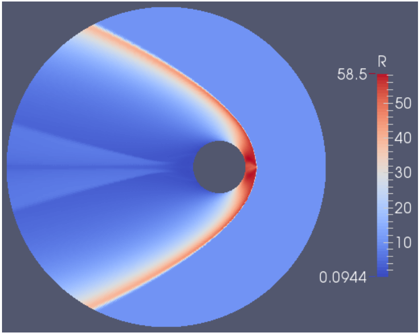
- An advantage of CDMATH is that the code can be run on very complex geometry build with Salome.



Flow around a cylinder

- An advantage of CDMATH is that the code can be run on very complex geometry build with Salome.





Final conclusion and perspective

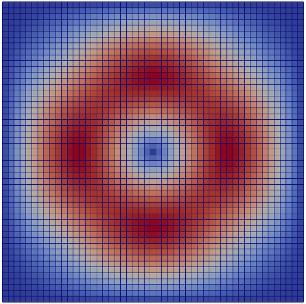
Conclusion :

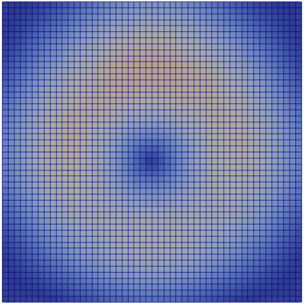
- We constat the accuracy at low Mach number of finite volume schemes on triangular meshes in the linear and non-linear cases.
- We constat the inaccuracy at low Mach number of the finite volume schemes on cartesian meshes in the linear and non-linear cases.
- The study of the linear case allows to propose a correction that gives an accurate scheme at low Mach number and gives the Godunov scheme when $M \geq 1$ on cartesian meshes.

Perpectives :

- Study the influence of the time discretization in the linear case.
- Extend the study to the barotropic Euler equation with porosity.

Thank you for your attention !





S. Dellacherie, P. Omnes. On the Godunov scheme applied to the variable cross-section linear equation. FVCA6, (4) :313–321, 2011.