A well-balanced scheme for two-fluid flows in spherical coordinates

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Outline

Model

We consider the Euler equations in spherical coordinates:

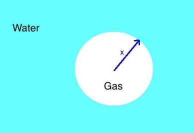
$$\begin{aligned}
\partial_t(A\rho) + \partial_x(A\rho u) &= 0,\\
\partial_t(A\rho u) + \partial_x(A(\rho u^2 + p)) &= p\partial_x A,\\
\partial_t(A\rho E) + \partial_x(A(\rho E + p)u) &= 0,\\
\partial_t(A\rho\varphi) + \partial_x(A\rho\varphi u) &= 0,
\end{aligned}$$

where A is the cross-section, ρ the density, u the velocity, E the total energy, φ the fraction of mass of gas and p satisfies the stiffened gas pressure law:

$$p(\rho, e, \varphi) = (\gamma(\varphi) - 1)\rho e - \gamma(\varphi)\pi(\varphi).$$

Applications:

- Spherical bubble:



- Nozzle:

Model

- Let $A_\triangle$ a piecewise constant approximation of A:

$$A_\triangle(x) = \sum_i A_i I_{]x_{i-1/2};x_{i+1/2}[}(x).$$

- The system becomes:

$$\begin{aligned}
\partial_t A_\triangle &= 0, \\
\partial_t(A_\triangle \rho) + \partial_x(A_\triangle \rho u) &= 0, \\
\partial_t(A_\triangle \rho u) + \partial_x(A_\triangle(\rho u^2 + p)) &= p\partial_x A_\triangle, \\
\partial_t(A_\triangle \rho E) + \partial_x(A_\triangle(\rho E + p)u) &= 0, \\
\partial_t(A_\triangle \rho\varphi) + \partial_x(A_\triangle \rho\varphi u) &= 0.
\end{aligned}$$

Model

We can write the system in the non-conservative form:

$$\partial_t W + \partial_x F(W) - S(W)\partial_x A_\triangle = 0,$$

where the vector of conservative variables is:

$$W = (A_\triangle, A_\triangle \rho, A_\triangle \rho u, A_\triangle \rho E, A_\triangle \rho \varphi)^T,$$

the conservative flux is:

$$F(W) = (0, A_\triangle \rho u, A_\triangle(\rho u^2 + p), A_\triangle(\rho E + p)u, A_\triangle \rho \varphi u)^T,$$

and the non-conservative source term is:

$$S\partial_x A_\triangle = (0, 0, p\partial_x A_\triangle, 0, 0).$$

Hyperbolicity

Theorem

The jacobian matrix of the non-conservative system:

$$\partial_t W + \partial_x F(W) - S(W)\partial_x A_\triangle = 0,$$

admits five real eigenvalues:

- $\lambda_0 = 0(LD)$,
- $\lambda_1 = u - c(GNL)$,
- $\lambda_2 = \lambda_3 = u(LD)$,
- $\lambda_4 = u + c(GNL)$.

Numerical scheme

- We propose a first order finite volume scheme with an Arbitrary Lagrangian Eulerian approach, the boundary $x^n_{i+1/2}$ can move at the speed of the fluid $v^n_{i+1/2}$ between t_n and t_{n+1}:

$$x^{n+1,-}_{i+1/2} = x^n_{i+1/2} + \Delta t v^n_{i+1/2}.$$

- We define an ALE numerical flux:

$$F(W_L, W_R, v^{\pm}) := F(W(W_L, W_R, v^{\pm})) - vW(W_L, W_R, v^{\pm})$$

where $W(W_L, W_R, v^{\pm})$ is obtained with an approximated Riemann solver described below.

Numerical scheme

The integration of $\partial_t W + \partial_x F(W) - S(W)\partial_x A_\triangle = 0$ on $]t_n; t_{n+1}[\times]x_{i-1/2}(t); x_{i+1/2}(t)[$ gives:

- if $v^n_{i+1/2} \leq 0$ and $v^n_{i-1/2} \geq 0$, the ALE scheme is:

$$\begin{aligned} \Delta x_i^{n+1} W_i^{n+1,-} - \Delta x_i^n W_i^n + \\ \Delta t \left(F(W_i^n, W_{i+1}^n, v_{i+1/2}^{n,-}) - F(W_{i-1}^n, W_i^n, v_{i-1/2}^{n,+}) \right) &= 0. \end{aligned}$$

- if $v^n_{i+1/2} > 0$, we have to add the following term on the left side of the equation above:

$$\Delta t \left(F(W_i^n, W_{i+1}^n, 0^-) - F(W_i^n, W_{i+1}^n, 0^+) \right),$$

- if $v^n_{i-1/2} < 0$, we add:

$$\Delta t \left(F(W_{i-1}^n, W_i^n, 0^-) - F(W_{i-1}^n, W_i^n, 0^+) \right).$$

Numerical scheme

To update $W_i^{n+1,-}$, we have to compute $F(W_L, W_R, v^{\pm})$. To do that, we need to:

- precise how we choose the ALE velocity v,
- explain how we compute $F(W_L, W_R, v^{\pm})$ for a given ALE velocity v.

Computing the interface speed v

When initial datas satisfy $\varphi \in \{0, 1\}$, the algorithm verifies

- if $\varphi_L = \varphi_R$, we take $v = 0$.
- if $\varphi_L \neq \varphi_R$, we use an exact Riemann solver in $(\varrho, u, p, \varphi)^T$ to compute u^* and p^*. We take $v = u^*$ and the ALE flux takes the Lagragian following form:

$$F(W_L, W_R, v^{\pm}) = (0, A^* p^*, A^* u^* p^*, 0, -A^* u^*)^T,$$

where

$$A^* = \begin{cases} A_L, & \textit{if} \quad v < 0, \\ A_R, & \textit{if} \quad v > 0. \end{cases}$$

Computing $W(W_L, W_R, 0^{\pm})$

Proposition

The Riemann invariants associated to the stationnary wave $\lambda_0 = 0$ are:

- φ,
- $s = (p + \pi(\varphi))\rho^{-\gamma(\varphi)}$,
- $Q = \rho A u$,
- $H = E + \frac{p}{\rho}$.

Considering the set of variables $Z = (A, \varphi, s, Q, H)^T$, the system $\partial_t W + \partial_x F(W) - S(W)\partial_x A_\triangle = 0$ becomes:

$$\partial_t Z + C(Z)\partial_x Z = 0.$$

VFRoe-ncv solver

We want to solve

$$\partial_t Z + C(Z)\partial_x Z = 0,$$

with the initial datas

$$Z(x,0) = \begin{cases} Z(W_L) & , \ x < 0 \\ Z(W_R) & , \ x > 0 \end{cases}.$$

- We linearize $C(Z)$ and we solve the linear Riemann problem:

$$\partial_t Z + C(\hat{Z})\partial_x Z = 0,$$

  with the same initial datas.
- We obtain $Z(W_L, W_R, 0^{\pm})$.

Property of $Z(W_L, W_R, 0^{\pm})$

Proposition

The interface Riemann solver computes intermediate states $Z(W_i, W_{i+1}, 0^{\pm})$ which verify:

- $A^-_{i+1/2} = A_i$,
- $\varphi^-_{i+1/2} = \varphi^+_{i+1/2}$,
- $s^-_{i+1/2} = s^+_{i+1/2}$,
- $A^+_{i+1/2} = A_{i+1}$,
- $Q^-_{i+1/2} = Q^+_{i+1/2}$,
- $H^-_{i+1/2} = H^+_{i+1/2}$.

$Z(W_L, W_R, 0^\pm) \rightarrow W(W_L, W_R, 0^\pm)$

Problem

The map $W \mapsto Z$ is not bijective.

To obtain $\rho^-_{i+1/2}$, we need to solve:

$$L(\rho) = \frac{\gamma(\varphi_{i+1/2})}{\gamma(\varphi_{i+1/2}) - 1} s_{i+1/2}\rho^{\gamma(\varphi_{i+1/2})-1} + \frac{Q^2_{i+1/2}}{2\rho^2(A^-_{i+1/2})^2} - H_{i+1/2} = 0.$$

and we take:

- $u^-_{i+1/2} = \dfrac{Q_{i+1/2}}{\rho^-_{i+1/2}A^-_{i+1/2}}$,
- $p^-_{i+1/2} = s_{i+1/2}(\rho^-_{i+1/2})^{\gamma(\varphi_{i+1/2})} - \pi(\varphi_{i+1/2})$.

$\Longrightarrow W(W_L, W_R, 0^\pm)$ is computed.

Entropy correction

Definition

If $\lambda_k(W_i^n) \leq 0 \leq \lambda_k(W_{i+1}^n)$ in a non linear k-field, then the numerical flux F is replaced by:

$$\begin{aligned} G(W_i^n, W_{i+1}^n, 0^{\pm}) &= F(W_i^n, W_{i+1}^n, 0^{\pm}) \\ &- \min_k(|\lambda_k(W_i^n)|, |\lambda_k(W_{i+1}^n)|)(W_{i+1}^n - W_i^n)/2. \end{aligned}$$

Random sampling remap

We go back to the original grid by the Glimm procedure. We construct a sequence of pseudo-random $\omega_n \in [0,1[$, and we take:

$$W_i^{n+1} = \begin{cases} W_{i-1}^{n+1,-}, & \textit{if} \quad \omega_n < \frac{\Delta t^n}{\Delta x_i} \max(v_{i-1/2}^n, 0), \\ W_i^{n+1,-}, & \textit{if} \quad \frac{\Delta t^n}{\Delta x_i} \max(v_{i-1/2}^n, 0) \leq \omega_n \\ & \textit{and if} \quad \omega_n \leq 1 + \frac{\Delta t^n}{\Delta x_i} \min(v_{i+1/2}^n, 0), \\ W_{i+1}^{n+1,-}, & \textit{if} \quad \omega_n > 1 + \frac{\Delta t^n}{\Delta x_i} \min(v_{i+1/2}^n, 0). \end{cases}$$

Properties of the scheme

The constructed scheme has the following properties:

- it is well-balanced, meaning that **it preserves exactly all stationary states** (i.e. initial datas for which the quantities φ, s, Q, H are constant);
- if the fraction of gas φ takes only the two values 0 or 1, this property is exactly preserved at any time.

Stationary contact

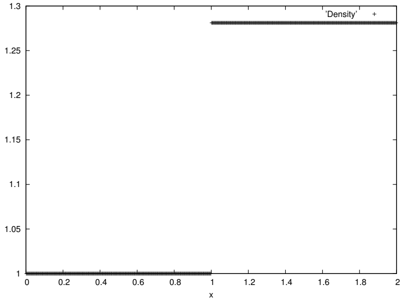
We choose the initial datas:

Quantities	L.	Right
$\rho(kg.m^{-3})$	1	1.281128576
$u(m.s^{-1})$	1	0.5203745191
$p(Pa)$	1	1.414587466
φ	1	1
A	1	1.5

such that:

- $\varphi_L = \varphi_R$,
- $s_L = s_R$,
- $Q_L = Q_R$,
- $H_L = H_R$.

We obtain at $t = 0.2s$:

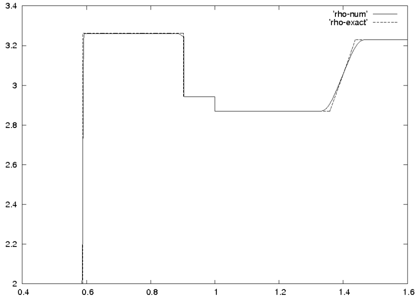


Comparison with an exact solution

The initial datas are:

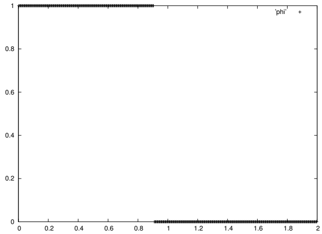
Quantities	L.	Right
$\rho(kg.m^{-3})$	2	3.230672602
$u(m.s^{-1})$	0.5	-0.4442565900
$p(Pa)$	1	12
φ	1	0
A	1.5	1
γ	1.4	1.6
$\pi(Pa)$	0	2

We obtain at $t = 0.2s$:

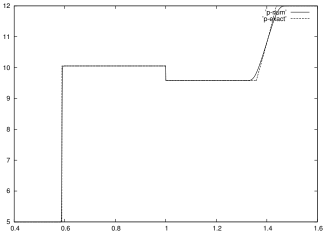


Comparison with an exact solution

- There is no diffusion on φ:



- There is no jump on velocity and pressure:



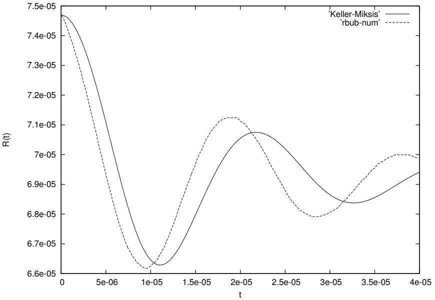
Implosion of a bubble

We simulate the collapse of a spherical bubble of gas in liquid water. Initial datas are:

Quantities	L.	Right
$\rho(kg.m^{-3})$	0.92	1000
$u(m.s^{-1})$	0	0
$p(Pa)$	72567.68	10^5
φ	1	0
A	1.5	1
γ	1.4	3
$\pi(Pa)$	0	733333.33

with $A(x) = 4\pi x^2$.

We plot the radius of the bubble as compared to the EDO model of Keller-Miksis, we obtain:



Conclusion

- We have constructed a new well-balanced scheme which does not diffuse the interface.
- We will pass this algorithm on GPU.
- We will test our algorithm for flow in a nozzle in one or two dimensions.
- We will compare this algorithm with other algorithms.

Thank you for your attention!