A well-balanced scheme for two-fluids flows in variable cross-section ducts

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Model

We consider the flow of a mixture of two compressible fluids (a gas and a liquid, for instance) in a cross-section duct. The model is the following:

$$\partial_t W + \partial_x F(W) = S(W) \tag{1}$$

where the vector of conservative variables is:

$$W = (A\rho, A\rho u, A\rho E, A\rho\varphi, A)^T, \tag{2}$$

the conservative flux is:

$$F(W) = (A\rho u, A(\rho u^2 + p), A(\rho E + p)u, A\rho\varphi u, 0)^T, \tag{3}$$

and the non-conservative source term is:

$$S = (0, p\partial_x A, 0, 0, 0). \tag{4}$$

Without loss of generality, we consider a stiffened gas pressure law:

$$p(\rho, e, \varphi) = (\gamma(\varphi) - 1)\rho e - \gamma(\varphi)\pi(\varphi). \tag{5}$$

Numerical scheme

• Finite volume scheme

The system (**??**) is approched by a **finite volume scheme** on cells $]x_{i-1/2}, x_{i+1/2}[$, $i \in \mathbb{Z}$. We note τ the time step and $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ the size of the cell i. We note W_i^n the vector of conservative variables on cell i at time t_n. The cross-section A is approximated by a piecewise constant function, $A = A_i$ on cell i.
We consider an **Arbitrary Lagrangian Eulerian** (ALE) approach. The boundary $x_{i+1/2}^n$ can move at the speed $v_{i+1/2}^n$ between t_n and t_{n+1}:

$$x_{i+1/2}^{n+1,-} = x_{i+1/2}^n + \tau v_{i+1/2}^n. \tag{6}$$

We define an ALE numerical flux:

$$F(W_L, W_R, v^\pm) := F(W(W_L, W_R, v^\pm)) - vW(W_L, W_R, v^\pm) \tag{7}$$

where $W(W_L, W_R, v^\pm)$ is obtain with an approximated Riemann solver described in a next part.
If $v_{i+1/2}^n \leq 0$ and $v_{i-1/2}^n \geq 0$, the ALE scheme is:

$$\Delta x_i^{n+1} W_i^{n+1,-} - \Delta x_i^n W_i^n + \tau\left(F(W_i^n, W_{i+1}^n, v_{i+1/2}^{n,-}) - F(W_{i-1}^n, W_i^n, v_{i-1/2}^{n,+})\right) = 0. \tag{8}$$

To take into account the variable section, if $v_{i+1/2}^n > 0$ we have to add the following term on the left of the equation above

$$\tau\left(F(W_i^n, W_{i+1}^n, 0^-) - F(W_i^n, W_{i+1}^n, 0^+)\right), \tag{9}$$

and if $v_{i-1/2}^n < 0$, we add:

$$\tau\left(F(W_{i-1}^n, W_i^n, 0^-) - F(W_{i-1}^n, W_i^n, 0^+)\right). \tag{10}$$

• Computing the interface speed $v_{i+1/2}^n$

When initial datas satisfy $\varphi \in \{0, 1\}$, the algorithm verifies

- If we are not at the interface, i.e. if $\varphi_L = \varphi_R$, we take $v = 0$.
- If we are at the interface, i.e. if $\varphi_L \neq \varphi_R$, we use an exact Riemann solver to compute $u^*(W_L, W_R)$ and $p^*(W_L, W_R)$ which present no jump at the contact discontinuity. We take $v = u^*(W_L, W_R)$, $A^* = A_L$ if $v < 0$ and $A^* = A_R$ if $v > 0$. The ALE flux becomes Lagrangian and takes the following form:

$$F(W_L, W_R, v^\pm) = (0, A^* p^*, A^* u^* p^*, 0, -A^* u^*)^T. \tag{11}$$

[3] B. Keller and M. Miksis, *Bubble oscillations of large amplitude.*, J. Acoust. Soc. Am., 68(2) : 628-633, 1980.

• Well-balanced solver for $W(W_L, W_R, 0^\pm)$

As the Riemann invariants associated to the stationnary wave are φ, $s = (p + \pi(\varphi))\rho^{-\gamma(\varphi)}$, $Q = \rho A u$ and $H = E + \frac{p}{\rho}$, the idea is to use a **VFRoe scheme [1] in** $Z = (A, \varphi, s, Q, H)^T$.
We then obtain $Z(W_L, W_R, 0^\pm)$ from which we get $W(W_L, W_R, 0^\pm)$. In some cases, this change of variables is not bijective: we have to adapt our approach.

• Entropy correction

A VFRoe scheme approaches badly the waves of relaxation crossing the interface $x = 0$ and authorizes in these points non-physical shocks. It is the case when an eigenvalue associated to a non-linear k-field verifies the following inequalities: $\lambda_k(W_i^n) \leq 0 \leq \lambda_{k+1}(W_i^n)$.
We replace the numerical flux F by:

$$G(W_i^n, W_{i+1}^n, 0^\pm) = F(W_i^n, W_{i+1}^n, 0^\pm) - \min_k(|\lambda_k(W_i^n)|, |\lambda_k(W_{i+1}^n)|)(W_{i+1}^n - W_i^n)/2.$$

• Glimm remap

We go back to the original grid by the Glimm procedure [2]. We construct a sequence of pseudo-random $\omega_n \in [0, 1[$, and we take:

$$W_i^{n+1} = \begin{cases} W_{i-1}^{n+1,-}, \; si \; \omega_n < \frac{\tau_n}{\Delta x_i}\max(v_{i-1/2}^n, 0), \\ W_i^{n+1,-}, \; si \; \frac{\tau_n}{\Delta x_i}\max(v_{i-1/2}^n, 0) \leq \omega_n \leq 1 + \frac{\tau_n}{\Delta x_i}\min(v_{i+1/2}^n, 0), \\ W_{i+1}^{n+1,-}, \; si \; \omega_n > 1 + \frac{\tau_n}{\Delta x_i}\min(v_{i+1/2}^n, 0). \end{cases} \tag{12}$$

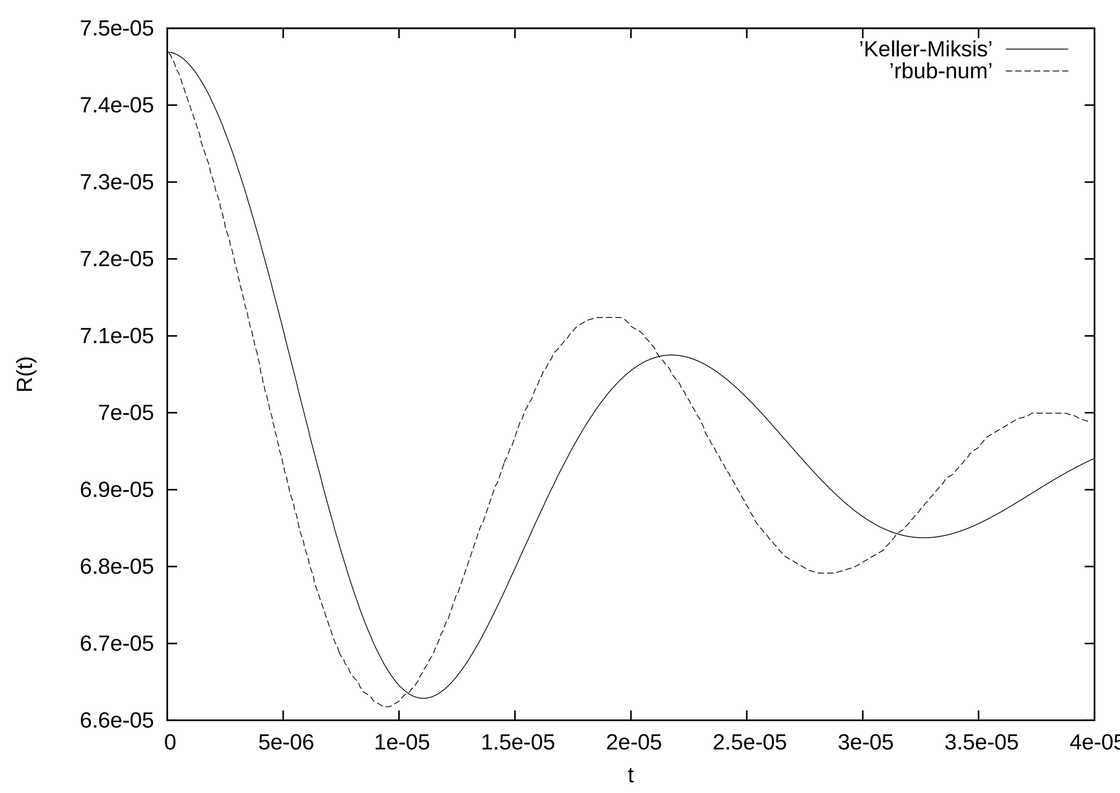
• Properties of the scheme

The constructed scheme has the following properties:

- it is well-balanced, meaning that **it preserves exactly all stationary states** (i.e. initial datas for which the quantities φ, s, Q, H are constant);
- if the fraction of gas φ takes only the two values 0 or 1, this property is exactly preserved at any time.

Implosion of a bubble

We valid our model by simulating the collapse of a spherical bubble of gas in liquid water. In that purpose, we assume an invariance under rotation and we adopt a 1D approach with a variable cross section $A(x) = 4\pi x^2$.
We plot the radius of the bubble that we compare to the ODE model of Keller-Miksis [3], we obtain:



Bibliography

[1] P. Helluy, J.-M. Hérard and H. Mathis, *A well-balanced approximate Riemann solver for variable cross-section compressible flows.*, Computational Fluid Dynamics, June 2009.

[2] C. Chalons and F. Coquel, *Computing material fronts with Lagrange-Projection approach.*, HYP2010 Proc. http://hal.archives-ouvertes.fr/hal-00548938/fr/.