Practical Work #4

The aim of this session is to apply the Finite Difference Method (FDM) to find approximate solutions to a Partial Differential Equations (PDE) problem and to cope with classical numerical issues.

1 Heat equation on a uniform grid (PDE)

We consider the PDE : Find ψ such that

$$\begin{cases} \frac{\partial \psi}{\partial \theta}(\theta,x) - \frac{\sigma^2}{2}\frac{\partial^2\psi}{\partial x^2}(\theta,x) = 0 & \forall(\theta,x)\in[0,T]\times[\underline{x},\overline{x}] \quad \text{(PDE)} \\ \psi(\theta,\overline{x}) = 0 & \forall\theta\in[0,T] \quad \text{(boundary condition in } x=\overline{x}) \\ \psi(\theta,\underline{x}) = Ke^{-(a+r)\theta-b\underline{x}} & \forall\theta\in[0,T] \quad \text{(boundary condition in } x=\underline{x}) \\ \psi(0,x) = e^{-bx}\max(0,K-e^x) & \forall x\in[\underline{x},\overline{x}] \quad \text{(initial condition in } \theta=0). \end{cases} \tag{1}$$

where ψ is a function of two variables θ and x.

Finding a solution (or an approximate solution) of (1) allows to obtain a solution (resp. approximate solution for the Black & Scholes equation thanks to the formula

$$P(T-\theta, e^x) = \psi(\theta,x)e^{a\theta+bx} \tag{2}$$

We set for the present study

$$K=100, \quad T=1, \quad \sigma=0.2, \quad r=0.04, \quad b=\frac{1}{2}-\frac{r}{\sigma^2}, \quad a=-r-\frac{\sigma^2 b^2}{2}, \quad \underline{x}=-50, \quad \overline{x}=50$$

and consider the discretization

$$\begin{cases} x_j = \underline{x}+(j-1)\Delta x, & 1\le j\le N_x \quad \Delta x = \frac{\overline{x}-\underline{x}}{N_x-1} \\ \theta_n = (n-1)\Delta\theta & 1\le n\le N_t \quad \Delta\theta = \frac{T}{N_t-1} \end{cases}$$

for some $N_x > 1$ and $N_t > 1$.

1. What are values of $\theta_1, \theta_{N_t}, x_1, x_{N_x}$?

2. The explicit Euler scheme writes

$$\frac{\psi_j^{n+1}-\psi_j^n}{\Delta\theta} - \frac{\sigma^2}{2}\left(\frac{\psi_{j+1}^n - 2\psi_j^n + \psi_{j-1}^n}{(\Delta x)^2}\right) = 0$$

adding discrete analogue of initial and boundary conditions (1) leads to the scheme

$$\begin{cases} \psi_j^{n+1} = \psi_j^n + \Delta\theta\frac{\sigma^2}{2}\left(\frac{\psi_{j+1}^n-2\psi_j^n+\psi_{j-1}^n}{(\Delta x)^2}\right) & 2\le j\le N_x-1, \quad 1\le n\le N_t-1 \quad \text{(Euler explicit scheme)} \\ \psi_1^n = Ke^{-(a+r)\theta_n - b\underline{x}} & 2\le n\le N_t \quad \text{(boundary condition in } x=\underline{x}) \\ \psi_{N_x}^n = 0 & 2\le n\le N_t \quad \text{(boundary condition in } x=\overline{x}) \\ \psi_j^1 = e^{-bx_j}\max(0,K-e^{x_j}) & 1\le j\le N_x \quad \text{(initial condition in } \theta=0) \end{cases} \tag{3}$$

Remark : for the sake of stability N_x and N_t must be chosen such that $\Delta\theta \le \frac{(\Delta x)^2}{\sigma^2}$.
Implement this scheme to compute ψ_j^n for all $1\le j\le N_x$ and $1\le n\le N_t$. Then plot the solution at time $\theta = T$ (ie plot $\psi_j^{N_t}$ as a function of x_j) for $N_x = 100, 1000, 10000$. What do you observe ?
As ψ_j^n is an approximation of $\psi(\theta_n, x_j)$, we have solved numerically problem (1).

3. Using (2), we define $S_j = e^{x_j}$ and $P_j^1 = \psi_j^{N_t} e^{aT+bx_j}$ for $1 \le j \le N_x$. Plot P_j^1 as a function of S_j. What do you observe ?

4. The implicit Euler scheme writes for $2 \le j \le N_x - 1, \quad 1 \le n \le N_t - 1$:

$$\frac{\psi_j^{n+1} - \psi_j^n}{\Delta\theta} - \frac{\sigma^2}{2}\left(\frac{\psi_{j+1}^{n+1} - 2\psi_j^{n+1} + \psi_{j-1}^{n+1}}{(\Delta x)^2}\right) = 0.$$

   or equivalently

$$\psi_j^{n+1}\left(1 + \frac{\Delta\theta\sigma^2}{(\Delta x)^2}\right) - \frac{\Delta\theta\sigma^2}{2(\Delta x)^2}\psi_{j+1}^{n+1} - \frac{\Delta\theta\sigma^2}{2(\Delta x)^2}\psi_{j-1}^{n+1} = \psi_j^n \qquad 2 \le j \le N_x - 1$$

   Initial and boundary conditions are the same as in (3). To use this scheme one need to solve a linear system of the form

$$A(\psi_j^{n+1})_{1\le j\le N_x} = (b_j^n)_{1\le j\le N_x}$$

   Write matrix $A \in \mathscr{M}_{N_x}(\mathbb{R})$ and vector $b^n \in \mathbb{R}^{N_x}$ using the implicit Euler scheme and boundary conditions of (3).
   Then implement the implicit Euler scheme to compute ψ_j^n for all $1 \le j \le N_x$ and $1 \le n \le N_t$. Finally, plot P_j^1 as a function of S_j.

5. The Crank-Nicholson scheme writes for $2 \le j \le N_x - 1$, $1 \le n \le N_t - 1$:

$$\frac{\psi_j^{n+1} - \psi_j^n}{\Delta\theta} - \frac{\sigma^2}{4}\left(\frac{\psi_{j+1}^{n} - 2\psi_j^{n} + \psi_{j-1}^{n}}{(\Delta x)^2}\right) - \frac{\sigma^2}{4}\left(\frac{\psi_{j+1}^{n+1} - 2\psi_j^{n+1} + \psi_{j-1}^{n+1}}{(\Delta x)^2}\right) = 0.$$

   Initial and boundary conditions are the same as in (3). To use this scheme one need to solve a linear system. Write this system and implement the Crank-Nicholson scheme. Then plot P_j^1 as a function of S_j.

6. compare performances of the explicit Euler scheme, implicit Euler scheme and Crank Nicholson scheme. N_x will be chosen equal to 100,1000 and 10000, and N_t so that the stability condition is verified.

2 Lorenz model (ODE)

The Lorenz model reads

$$\begin{cases} y_1' = 10(y_2 - y_1), & y_1(0) = -8, \\ y_2' = 28y_1 - y_2 - y_1 y_3, & y_2(0) = 8, \\ y_3' = y_1 y_2 - \frac{8}{3} y_3, & y_3(0) = 27. \end{cases} \tag{4}$$

No exact solution is known for this system of ODEs. That is why we use numerical schemes to construct an approximation of the solution.

1. Implement the explicit Euler scheme as well as the RK4 scheme.
2. Plot the graphs $t \mapsto \big(t, y_1(t)\big)$, $t \mapsto \big(t, y_2(t)\big)$ and $t \mapsto \big(t, y_3(t)\big)$ for both schemes.
3. Plot the curve $t \mapsto \big(y_1(t), y_2(t)\big)$.
4. Compare the numerical solutions. What can you say about the behaviour of the solution?