

Exercices

Exercise 1 Let (u_n) be the sequence defined by

$$u_0 = 0, \forall n \in \mathbb{Z}_+, u_{n+1} = \frac{u_n + 2}{4 - u_n}.$$

Set $v_n = \frac{u_n - 1}{u_n - 2}$ and $f(x) = \frac{x+2}{4-x}$.

1. Solve the equations $f(x) = 1$ and $f(x) = 2$.
2. Prove that if $u_n < 1$, then $u_{n+1} < 1$. Deduce that the sequence (u_n) is well defined.
3. Show that (v_n) is well defined and geometric. Deduce the expression of v_n with respect to n .
4. Derive the expression of u_n with respect to n . What is the limit of (u_n) ?

Exercise 2 Determine the expression of u_n solution of

$$\begin{cases} u_0 = 0, u_1 = 1, \\ u_{n+1} = -u_n - \frac{5}{36}u_{n-1}, n \geq 1. \end{cases}$$

Exercise 3 Write out the Cholesky algorithm. Determine the cost of the algorithm.

Exercise 4 Let A_α be the matrix

$$A_\alpha = \begin{pmatrix} \alpha & 0 & -1 \\ 0 & \alpha & -1 \\ -1 & -1 & \alpha \end{pmatrix}$$

parametrized by $\alpha \in \mathbb{R}$.

1. Determine the eigenvalues of A_α and some corresponding eigenvectors.
2. For which values of α the matrix A_α is invertible?
3. (a) For which values of α the matrix A_α admits a Cholesky decomposition?
 (b) For those values, compute the decomposition from the algorithm of Exercise 3.
 (c) By means of an up-down algorithm, solve explicitly the linear system $A_\alpha x = b$ for some $b \in \mathbb{R}^3$.
 (d) Deduce the expression of A_α^{-1} .
 (e) For a symmetric positive-definite matrix, the condition number of a matrix is the ratio of the largest eigenvalue by the lowest one. Compute the condition number of A_α .
 (f) Which asymptotics upon α corresponds to the most suitable case for inverting A_α ?
4. We assume in this question that $\alpha \neq 0$. We aim at approximating the solution of $A_\alpha x = b$ by means of an iterative method, which is the Jacobi method: writing $A_\alpha = D_\alpha - E - F$ where D_α is the diagonal matrix whose coefficients are the diagonal entries of A_α , E is lower triangular and F is upper triangular. The Jacobi method consists of the sequence $D_\alpha x^{n+1} = (E + F)x^n + b$.
 (a) Write down the matrices D_α , E and F .
 (b) For which values of α is the method convergent?
 (c) What are the induction relations for x_1^n , x_2^n and x_3^n .
 (d) Compute the expression of x^n with respect to n .
 (e) What is the limit of x^n as n goes to ∞ ? Comment.

Exercise 5 Give the solutions to the following ODEs:

$$\hat{y}'(t) = 2\hat{y}(t) + 1; \quad \hat{y}'(t) = \frac{t}{t^2 + 1}\hat{y}(t); \quad \hat{y}'(t) = -(\tan t)\hat{y}(t) + \cos t; \quad t\hat{y}'(t) = -\hat{y}(t) + t,$$

supplemented with the initial condition $\hat{y}(t_0) = y_0$. Specify for which t_0 such solutions exist.

Exercise 6 1. Determine the solutions to the second-order ODE

$$y''(t) - 2y'(t) + y(t) = 0.$$

2. We then focus on the ODE

$$\begin{cases} y''(t) - 2y'(t) + y(t) = \cos t, \\ y(0) = 0, \\ y'(0) = 1. \end{cases} \quad (1)$$

- Prove that there exists a unique maximal solution of (1).
- Prove that this solution is defined for $t \in \mathbb{R}$
- What are the eigenvalues of the corresponding matrix?
- Determine C such that $y(t) = C(t)e^t$ satisfies (1).
- Conclude.

Exercise 7 We consider the ordinary differential equation

$$(t^2 + 1)\hat{y}'(t) + t = t\hat{y}(t)^2. \quad (2)$$

- Determine two constant solutions of (2).
- Assume that $y(0) = y_0 \in]-1, 1[$. Prove the existence of a solution to (2) defined for $t \in \mathbb{R}$.
- Set $\hat{z} = \hat{y} - 1$. Show that \hat{z} satisfies

$$(t^2 + 1)\hat{z}'(t) = t(2\hat{z}(t) + \hat{z}^2(t)). \quad (3)$$

- Set $\hat{w} = \frac{1}{\hat{z}}$. Determine the equation satisfied by \hat{w} and solve (2).
- Apply the explicit and implicit Euler schemes to (2).

Exercise 8 These equations model the evolution of an isolated predator-prey system (for instance rabbits and lynx):

$$\begin{cases} x'(t) = x(t)(3 - y(t)), & x(0) = 1, \\ y'(t) = y(t)(x(t) - 2), & y(0) = 2. \end{cases} \quad (4)$$

- Determine which variable corresponds to the number of preys.
- Show that there is no constant solution to Problem (4).
- Rewrite Eqs. (4) as $\mathbf{Y}'(t) = \mathbf{F}(\mathbf{Y}(t))$, where $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$. Deduce that there exists a unique maximal solution \mathbf{Y} .
- Prove that x and y cannot vanish. Deduce the sign of each unknown.

5. We set $H(x, y) = x - 2 \ln x + y - 3 \ln y$. H is called the Hamiltonian of the system. Show that for all $t \geq 0$, $H(x(t), y(t)) = H(x(0), y(0))$.
6. Apply the explicit Euler scheme to Eq. (4). Is the numerical Hamiltonian also constant?
7. We introduce the symplectic Euler scheme:

$$\begin{cases} x_{n+1} = x_n + \Delta t x_{n+1} (3 - y_n), \\ y_{n+1} = y_n + \Delta t y_n (x_{n+1} - 2). \end{cases}$$

Prove this scheme is consistent.

In the sequel, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ denotes a smooth function of class \mathcal{C}^1 . We aim at approximating the solution of the ODE

$$\hat{y}'(t) = f(t, \hat{y}(t)). \quad (5)$$

Let T be some positive number and $N \in \mathbb{Z}_+$, $N \neq 0$. Then we set $\Delta t = \frac{T}{N}$ and $t^n = n\Delta t$, $0 \leq n \leq N$.

Exercise 9 We assume in this exercise that $f(t, y) = -y$.

1. Solve (5) in this case supplemented with the initial condition $\hat{y}(0) = 1$.
2. Apply the explicit Euler scheme to construct the sequence (y_n) .
3. Yield the explicit expression of y_n with respect to n and Δt . Is the scheme relevant for any Δt ?
4. Compare y_N and $\hat{y}(T)$. Conclude.
5. Follow the same directions about the Heun scheme.
6. Which scheme seems to be the most efficient?

Exercise 10 We take in this exercise $f(t, y) = 1 - 2y$.

1. Solve (5) in this case supplemented with the initial condition $\hat{y}(0) = 1$.
2. Apply the implicit Euler scheme to construct the sequence (y_n) .
3. Yield the explicit expression of y_n with respect to n and Δt .

Exercise 11 To provide an approximate solution to (5), we propose the scheme

$$\frac{3y_{n+2} - 4y_{n+1} + y_n}{2} = \Delta t f(t^{n+2}, y_{n+2}).$$

1. How can this scheme be initialized?
2. Show that the scheme is convergent. Determine its order.
3. Is this scheme explicit?
4. In the case $f(t, y) = -y$, solve the linear inductive relation for y_n .
5. Propose a modification of the right hand side in the previous scheme to improve the order.

Exercise 12 *The enhanced Euler scheme reads*

$$y_{n+1} = y_n + \Delta t f \left(t^n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} f(t^n, y_n) \right).$$

1. Compute $\hat{y}''(t)$ for \hat{y} solution of (5).
2. Show this scheme is convergent and determine its order.

Exercise 13 *We aim at studying a numerical scheme dedicated to the resolution of the autonomous ordinary differential equation*

$$\begin{cases} y'(t) = f(y(t)), & (6a) \\ y(0) = y_0, & (6b) \end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a \mathcal{C}^∞ function. We consider the lintrap scheme

$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_n) + f'(y_n) \frac{y_{n+1} - y_n}{2} \tag{7}$$

where $\Delta t > 0$ is some positive number and we set $t^n = n\Delta t, n \geq 0$.

1. Does there exist a unique solution to ODE (6)?
2. **Study of the numerical scheme**
 - (a) Is the scheme explicit or implicit?
 - (b) Is the scheme well-defined in any case? Show that if f is monotone-decreasing, then the scheme is well-defined.
 - (c) For \hat{y} solution to (6), compute $\hat{y}''(t)$.
 - (d) Prove that (7) is a consistent scheme up to order 2.
3. **Investigation of a particular case.** We suppose in this question that $f(y) = (y + 1)^2$.
 - (a) Compute the exact solution \hat{y} to (6) in that case.
 - (b) Apply Scheme (7) to ODE (6). Express y_{n+1} as a function of Δt and y_n .
 - (c) Let us introduce $z_n = \frac{1}{y_n + 1}$. Show that (z_n) satisfies an arithmetic progression.
 - (d) Deduce the expression of y_n with respect to n and Δt .
 - (e) Compare y_n and $\hat{y}(t^n)$. Conclude.

Exercise 14 *Let us consider the system*

$$\begin{cases} x'(t) = y(t), & x(0) = 1, \\ y'(t) = -x(t), & y(0) = 0. \end{cases} \tag{8}$$

1. Prove that the trajectories $t \mapsto (x(t), y(t))$ are included in the unit circle $x^2 + y^2 = 1$.
2. Write out the explicit Euler scheme, the implicit Euler scheme and the Crank-Nicholson scheme for the resolution of (8).
3. Do these schemes preserve the trajectories?

Exercise 15 Let us consider the ODE

$$\begin{cases} \hat{y}'(t) + 3\hat{y}(t)^2 = 0, \\ \hat{y}(0) = 1. \end{cases} \quad (9)$$

1. Solve (9). What is the limit of $\hat{y}(t)$ as $t \rightarrow +\infty$?
2. Apply the explicit Euler scheme to this ODE and study the limit as $n \rightarrow +\infty$.
3. We propose the following scheme

$$\frac{y_{n+1} - y_n}{\Delta t} + 3\left(\frac{y_{n+1} + y_n}{2}\right)^2 = 0. \quad (10)$$

- (a) Derive an expression of y_{n+1} with respect to y_n and deduce a condition upon Δt for the limit to be correct as $n \rightarrow +\infty$.
- (b) Show this scheme is consistent at order 2.

Exercise 16 Let us consider the following differential equation

$$-u''(x) + u'(x) + \left(\alpha^2 - \frac{1}{4}\right)u(x) = f(x), \quad (11)$$

where $\alpha > 0$ is some real number and f is a continuous function over \mathbb{R}_+ . To supplement Equation (11), we propose two types of boundary conditions:

$$u(0) = 0, \quad u'(0) = 1, \quad (\text{BC1})$$

$$u(0) = 0, \quad u(1) = 2. \quad (\text{BC2})$$

1. **We assume in this question ONLY that $f(x) = 0$ for all $x \geq 0$.**
 - (a) Compute the expression of the solution to (11) together with (BC1).
 - (b) What is the solution for (BC2)?
2. **General case.**
 - (a) Prove that there exists a unique solution to Equation (11) (for some given f) supplemented with (BC1).
 - (b) What can we say about the problem (11)–(BC2)?
 - (c) Let us set

$$\forall x \geq 0, \hat{u}(x) = e^{(\frac{1}{2}+\alpha)x} \left(c_0 - \frac{1}{2\alpha} \int_0^x f(y) e^{-(\frac{1}{2}+\alpha)y} dy \right) + e^{(\frac{1}{2}-\alpha)x} \left(d_0 + \frac{1}{2\alpha} \int_0^x f(y) e^{-(\frac{1}{2}-\alpha)y} dy \right).$$
 Show that \hat{u} satisfies (11).
 - (d) Determine (c_0, d_0) so that \hat{u} also satisfies (BC1). Same question for (BC2).
 - (e) Is this expression for \hat{u} always useful?
3. **Numerical approach.** Let us set $\Delta x = \frac{1}{N-1}$ for some integer $N \geq 2$ and $x_i = (i-1)\Delta x$ for $i \in \{1, \dots, N\}$. In this section, we are interested in designing a numerical scheme to provide approximations u_i of $\hat{u}(x_i)$.
 - (a) Propose a finite-difference scheme to approximate the solution to (11).
 - (b) How to take (BC1) into account? Write out the corresponding algorithm to compute u_i for all $i \in \{1, \dots, N\}$.

(c) Same question for (BC2). What can you say about the matrix of the underlying linear system?

4. **Substitution.** Let v be the function such that $u(x) = v(x)e^{x/2}$.

(a) Prove that v is a solution of the following equation

$$-v''(x) + \alpha^2 v(x) = f(x)e^{-x/2}. \tag{12}$$

(b) What are the boundary conditions for v corresponding to (BC2)?

(c) Propose a finite-difference numerical scheme to solve (12)–(BC2). Does this seem more practical than in Question 3.(c)?

(d) We admit that the Cholesky factorization of a tridiagonal matrix $A \in \mathcal{M}_n(\mathbb{R})$ is $B^T B$ where B is a bidiagonal upper matrix of the form

$$\begin{pmatrix} \sqrt{\beta_1} & \gamma_2 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma_{n-1} \\ 0 & \cdots & 0 & \sqrt{\beta_n} \end{pmatrix}$$

Adapt the Cholesky algorithm to the factorization of the matrix of Question 4.(c). In particular, show that (β_i) satisfies the inductive relation

$$\beta_i + \frac{1}{\beta_{i-1}} = 2 + \alpha^2 \Delta x^2, \quad \beta_1 = 2 + \alpha^2 \Delta x^2.$$

What is γ_i equal to?

Exercise 17 We aim at solving the autonomous (i.e. f does not depend on t) ordinary differential equation

$$\begin{cases} y'(t) = f(y(t)), \\ y(0) = \frac{1}{2}. \end{cases} \tag{13}$$

We assume that f is of class $\mathcal{C}^2(\mathbb{R})$.

1. Justify that ODE (13) has a unique equation denoted \hat{y} . What is the regularity of \hat{y} ?
2. Give the expression of the solution in the following cases:
 - (a) $f(x) = 1$;
 - (b) $f(x) = \lambda x$ with $\lambda \in \mathbb{R}$.
3. In the general case, we cannot provide an explicit expression. That is why we aim at constructing approximate values of the solution at some points. More precisely, we set

$$t^n = n\Delta t, \Delta t = \frac{3}{N},$$

for some fixed integer $N \geq 1$. A numerical scheme is a method whose purpose is to compute y_n which is an approximation of $\hat{y}(t^n)$.

- (a) Do we have $y_n = \hat{y}(t^n)$?
- (b)
 - i. Apply the explicit Euler scheme to ODE (13). Express y_{n+1} as a function of y_n .
 - ii. How many values do we need in order to initialize the sequence (y_n) ?

- iii. Write out the algorithm leading to the computation of the sequence (y_n) .
- iv. Recall the order of this scheme.

(c) We are interesting in the multi-step scheme

$$z_{n+3} - z_{n+1} = \Delta t \left(\frac{7}{3} f(z_{n+2}) - \frac{2}{3} f(z_{n+1}) + \frac{1}{3} f(z_n) \right). \quad (14)$$

- i. Prove the consistency of Scheme (14).
- ii. Study its stability.
- iii. Deduce that this scheme is convergent.
- iv. Is this scheme explicit or implicit? Justify your answer.
- v. Determine the order of Scheme (14).
- vi. How many values do we need in order to initialize the sequence (z_n) ? Explain how to compute these initializing values.
- vii. Write out the algorithm leading to the computation of the sequence (z_n) .
- viii. Which scheme would you recommend: Euler (Q. 3.(b)) or Scheme (14)?
- ix. Apply Scheme (14) when $f(x) = 1$. Compute the exact expression of z_n for all n .

Exercise 18 We focus in this exercise on the ordinary differential equation

$$\begin{cases} x_1'(t) = -x_1(t) - x_1(t)x_2(t), & x_1(0) = \frac{1}{3}, \\ x_2'(t) = -\frac{x_2(t)}{x_1(t)}, & x_2(0) = \frac{2}{3}. \end{cases} \quad (15)$$

We set $F(x, y) = \begin{pmatrix} -x - xy \\ -\frac{y}{x} \end{pmatrix}$ and $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

1. Rewrite ODE (15) by means of X and F .
2. Does there exist a solution to ODE (15)?
3. Is it possible to provide an explicit expression of the solution?
4. Let $\Delta t > 0$ be such that $\Delta t < \frac{1}{2}$. We propose the numerical scheme

$$\begin{cases} \frac{x_1^{n+1} - x_1^n}{\Delta t} = -x_1^n - x_1^n x_2^{n+1}, & x_1^0 = \frac{1}{3}, \\ \frac{x_2^{n+1} - x_2^n}{\Delta t} = -\frac{x_2^{n+1}}{x_1^n}, & x_2^0 = \frac{2}{3}. \end{cases} \quad (16)$$

- (a) Show by induction that the sequences (x_1^n) and (x_2^n) belong to $(0, 1)$ and are monotone-decreasing.
- (b) Deduce that they are convergent. Determine their limits.
- (c) Is Scheme (16) explicit?
- (d) Express x_1^{n+1} and x_2^{n+1} as functions of x_1^n , x_2^n and Δt . Deduce that this one-step scheme is consistent.

Exercise 19 We now study the pure advection equation

$$\begin{cases} \partial_t Y(t, x) + \alpha \partial_x Y(t, x) = 0, & x \in (0, 1), \\ Y(t, 0) = 0, \\ Y(0, x) = Y_0(x), \end{cases} \quad (17)$$

with the same assumptions for α and Y_0 as in the previous exercise.

1. Let Y_1 and Y_2 be two smooth solutions of PDE (17). Show that $Y_1 = Y_2$ by using

$$E(t) = \int_0^1 |Y_1(t, x) - Y_2(t, x)|^2 dx.$$

2. Deduce that function \hat{Y} defined by

$$\hat{Y}(t, x) = \begin{cases} Y_0(x - \alpha t), & \text{if } x - \alpha t \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is the unique smooth solution of (17).

3. We now aim at studying numerical schemes simulating PDE (17). To do so, let us introduce $N_x \geq 2$ an integer and $\Delta t > 0$ a real number. We then set

$$\Delta x = \frac{1}{N_x - 1}, \quad x_i = (i - 1)\Delta x, \quad 1 \leq i \leq N_x, \quad \text{and} \quad t^n = (n - 1)\Delta t, \quad n \geq 1.$$

We propose the following schemes

$$\frac{Y_i^{n+1} - Y_i^n}{\Delta t} + \alpha \frac{Y_i^n - Y_{i-1}^n}{\Delta x} = 0, \tag{18}$$

$$\frac{Y_i^{n+1} - Y_i^n}{\Delta t} + \alpha \frac{Y_i^{n+1} - Y_{i-1}^{n+1}}{\Delta x} = 0. \tag{19}$$

- (a) **Upwind scheme** (18): write out the algorithm corresponding to the computation of Y_i^n for all $i \in \{1, \dots, N_x\}$ and $n \geq 1$.
- (b) **Upwind scheme** (18): show that this scheme is consistant and using Exercise 20, derive a stability condition.
- (c) **Implicit scheme** (19): write out the corresponding algorithm. Does it require the resolution of a linear system?

Exercise 20 Let us study the 1D advection–diffusion equation

$$\begin{cases} \partial_t Y(t, x) + \alpha \partial_x Y(t, x) - \nu \partial_{xx}^2 Y(t, x) = 0, & x \in (0, 1), \\ Y(t, 0) = Y(t, 1) = 0, \\ Y(0, x) = Y_0(x), \end{cases} \tag{20}$$

for some constant velocity field $\alpha > 0$ and constant diffusion coefficient $\nu \neq 0$. The initial datum Y_0 is assumed to be smooth.

- 1. Propose a discretization of (20) inspired by the previous implicit scheme. Does it require the resolution of a linear system? If so, what can you say about the matrix?
- 2. Let us set $Z(t, x) = Y(t, x) \exp\left[\frac{-\alpha}{4\nu}(2x - \alpha t)\right]$. Show that Z satisfies the following PDE

$$\partial_t Z - \nu \partial_{xx}^2 Z = 0 \tag{21}$$

with suitable initial and boundary conditions. Do you think it is more relevant to discretize the equivalent equation (21)?