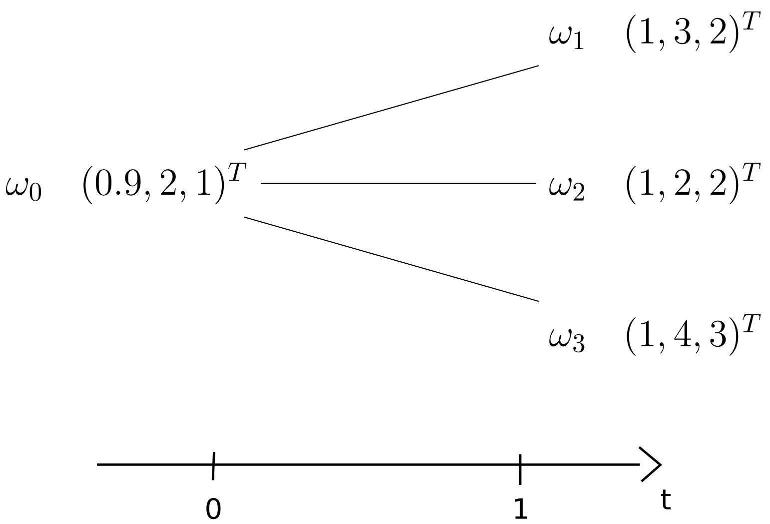
Advanced Algorithms for Finance
Master of Financial Engineering - M2
Intermediate Exam - Duration : 1 hour

No documents - <u>You can use a calculator</u>

Please report clear and detailed answers explicitly, as well as all the formulas you use and all the theorems you invoke.

Problem 1 : We consider the following model over one period, made up of three securities numbered $i = 0$, 1, and 2 :



The vectors given in this figure provide the values of the three securities in every state of the model.

1) The model is not binomial. Indeed, ω_0 is followed by three states instead of two at time $t = 1$.
2) Yes, the security $i = 0$ is a numeraire because

$$\forall j \in [\![1, 3]\!], \quad S_1^{(0)}(\omega_j) = 1 > 0.9.$$

The risk-free rate r is

$$1 + r = \frac{1}{0.9} \Rightarrow r = 1.1111 - 1 \Rightarrow r = 0.1111.$$

3) Yes, it is attainable. A security $X \in \mathbb{R}^3$ (row vector) is attainable if there exists $\theta \in \mathbb{R}^3$ (row vector) such that

$$X = \theta S_1 \Leftrightarrow X^T = S_1^T \theta^T$$

where

$$S_1 = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

We use a Gauss algorithme to found θ by solving

$$\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 4 & 3 & 7 \end{array}$$

With different operation on the line, we obtain

$$\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}$$

Then, $X = [4, 5, 7]$ is attainable with portfolio $\theta = (-1, -1, 4)$.

4) Yes, the market is complete because

$$rank(S_1) = 3.$$

5) It is sufficient to prove that there is no arbitrage of typa A. We know that θ is an arbitrage of type A if

$$\begin{cases} \theta S_0 < 0 \\ \theta S_1 = 0. \end{cases}$$

However,

$$\theta S_1 = 0 \Leftrightarrow S_1^T \theta^T = 0 \Leftrightarrow \theta^T \in Ker(S_1^T) = \{0\}$$

because $rank(S_1) = 3$. Then $\theta = (0, 0, 0)$ and it is not possible to have $\theta S_0 < 0$. Then, there is no arbitrage of type A and linear pricing hypothesis is satisfied.

Further assume that the linear pricing hypothesis is satisfied.

5) Under linear pricing hypothesis, the value at time $t = 0$ of the security is

$$\theta S_0 = (2, 1, 3) \begin{pmatrix} 0.9 \\ 2 \\ 1 \end{pmatrix} = 2 \times 0.9 + 2 + 3 = 6.8.$$

6) Since linear pricing hypothesis hold, we have

$$\pi = (\pi_1, \pi_2, \pi_3)^T \text{ is a vector of state prices} \Leftrightarrow \begin{cases} (\pi_1, \pi_2, \pi_3) > 0, \\ S_1(\pi_1, \pi_2, \pi_3)^T = S_0. \end{cases}$$

We use Gauss algorithme to solve $S_1(\pi_1, \pi_2, \pi_3)^T = S_0$. We want solve

$$\left.\begin{array}{ccc|c} 1 & 1 & 1 & 0.9 \\ 3 & 2 & 4 & 2 \\ 2 & 2 & 4 & 1 \end{array}\right. .$$

With some operations on the line we obtain

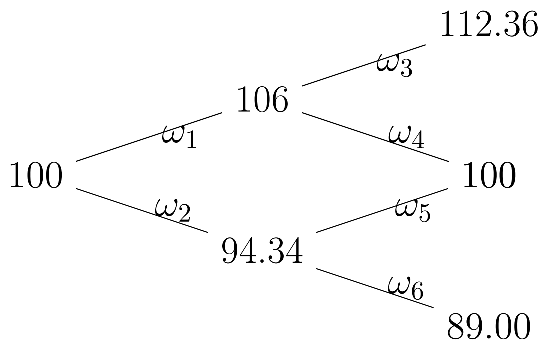
$$\begin{array}{ccc|c} 1 & 0 & 0 & 1.8 \\ 0 & 1 & 0 & -0.1 \\ 0 & 0 & 1 & -0.8 \end{array}$$

and since there is two negative components, there is no vector of state prices.

7) and 8) By applying the Fundamental theorem of asset pricing (part 1), we know that there does not exist a risk-neutral probability distribution $\mathcal{Q}$ and that there exist an arbitrage portfolio θ.

Problem 2 :

1) We have



2) $\mathcal{Q} = (q_1, q_2)$ is a risk-neutral probability distribution if
— $q_1 > 0$ and $q_2 > 0$,
— $q_1 + q_2 = 1$,
— the martingale property is satisfied

$$\forall t \geq 0, \quad \frac{S_t}{S_t^{(0)}} = \mathbb{E}_t^{\mathcal{Q}}\left(\frac{S_{t+1}}{S_{t+1}^{(0)}}\right).$$

Then, we can note $\mathcal{Q} = (q, 1-q)$ with $0 < q < 1$ and

$$\begin{aligned} \forall t \geq 0, \quad \frac{S_t}{S_t^{(0)}} &= \mathbb{E}_t^{\mathcal{Q}}\left(\frac{S_{t+1}}{S_{t+1}^{(0)}}\right) \\ &= q\frac{u\,S_t}{S_{t+1}^{(0)}} + (1-q)\frac{d\,S_t}{S_{t+1}^{(0)}} \end{aligned}$$

where $S_t^{(0)} = (1+r)^t$, then

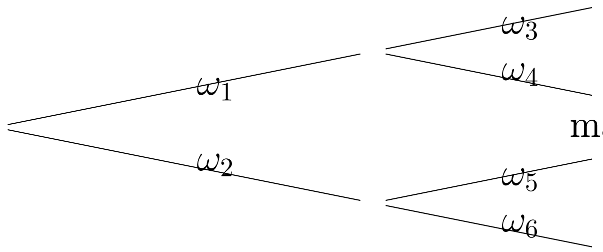
$$\begin{aligned} S_t &= q\frac{u\,S_t}{1+r} + (1-q)\frac{d\,S_t}{1+r} \\ \Longrightarrow 1+r &= qu + (1-q)d = q\,(u-d) + d \\ \Longrightarrow q &= \frac{1+r-d}{u-d}. \end{aligned}$$

Since $d < 1+r < u$, we have $0 < q < 1$.
In our case,

$$q = 0.6570.$$

3) A call option is a security that gives its owner the right to buy a given security (called the "underlying" security) at a date $T > 0$ (exercise date) for a certain price (called "strike") determined at time $t = 0$.
4) A european call option is a call whose exercise date $t = T > 0$ is determined at time $t = 0$. The date T is also called expiration date.
5) We compute the payoff at exercice date :

$\max(112.36 - 105, 0) = 7.36$

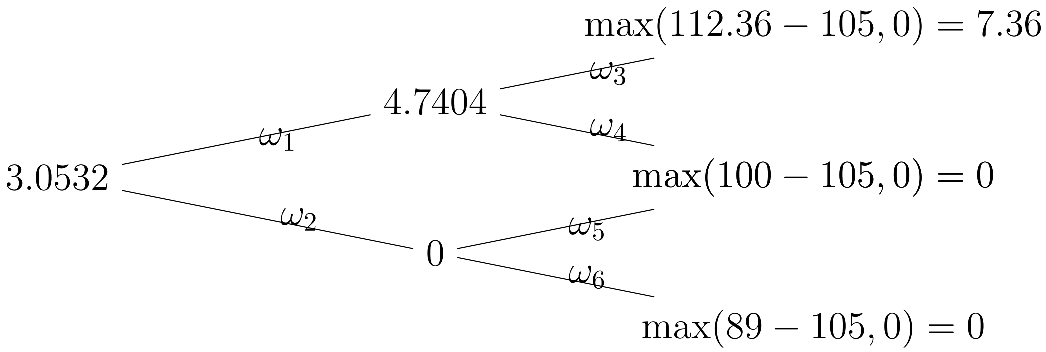


$\max(100 - 105, 0) = 0$

$\max(89 - 105, 0) = 0$

We ompute the value of the option for every node of the model (from $t = 2$ to $t = 0$, that is, from right to left in the tree) under the assumption that all deflated prices are martingales. We have

$$\begin{aligned} C_1(\omega_2) &= \frac{\mathbb{E}_1^{\mathcal{Q}}\left(\max\left(S_2^{(1)} - K, 0\right)\right)}{1+r} \\ &= \frac{q \times 7.36 + (1-q) \times 0}{1+r} \\ &= 4.7404 \end{aligned}$$



The price at time $t = 0$ of a european call with strike $K = 95$ is

$$C = 3.0532.$$