Advanced Algorithms for Finance
Master of Financial Engineering - M2
Final Exam 2014/2015 - Duration : 2h
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No documents - <u>You can use a calculator</u>

> Please report clear and detailed answers explicitly, as well as all the formulas you use and all the theorems you invoke.

Problem 1 :

1) Yes, the security $i = 0$ is a numeraire because

$$\forall j [\![1, 4]\!], \quad S_1^{(0)}(\omega_j) = 1.05 > 1.$$

The risk-free rate r is

$$(1 + r) \times 1 = 1.05 \Rightarrow r = 0.05 = 5\%.$$

2) The security

$$X = (11.05, 13.05, 15.05, 15.05)$$

is attainable if there exists $\theta = (\theta_0, \theta_1, \theta_3) \in \mathbb{R}^3$ such that

$$X = \theta S_1 \Leftrightarrow S_1^T \theta^T = X^T$$

where

$$S_1 = \begin{pmatrix} 1.05 & 1.05 & 1.05 & 1.05 \\ 4 & 2 & 5 & 3 \\ 1 & 4 & 2 & 4 \end{pmatrix}.$$

We need to solve

$$\begin{pmatrix} 1.05 & 4 & 1 \\ 1.05 & 2 & 4 \\ 1.05 & 5 & 2 \\ 1.05 & 3 & 4 \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 11.05 \\ 13.05 \\ 15.05 \\ 15.05 \end{pmatrix}$$

which we can write as

$$\begin{array}{ccc|c} 1.05 & 4 & 1 & 11.05 \\ 1.05 & 2 & 4 & 13.05 \\ 1.05 & 5 & 2 & 15.05 \\ 1.05 & 3 & 4 & 15.05 \end{array} \qquad \begin{array}{l} \\ L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1.05 & 4 & 1 & 11.05 \\ 0 & -2 & 3 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & 3 & 4 \end{array}\right) \quad \begin{array}{l} \\ \\ L_3 \leftarrow 2L_3 + L_2 \\ L_4 \leftarrow 2L_4 - L_2 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1.05 & 4 & 1 & 11.05 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 3 & 6 \end{array}\right) \quad \begin{array}{l} \\ \\ L_3 \leftarrow L_3/5 \\ L_4 \leftarrow L_4/3 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1.05 & 4 & 1 & 11.05 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} \\ L_2 \leftarrow L_2 - 3L_3 \\ \\ \\ \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1.05 & 4 & 1 & 11.05 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} \\ L_2 \leftarrow L_2/(-2) \\ \\ \\ \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1.05 & 4 & 1 & 11.05 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} L_1 \leftarrow L_1 - 4L_2 - L_3 \\ \\ \\ \\ \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1.05 & 0 & 0 & 1.05 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} L_1 \leftarrow L_1/1.05 \\ \\ \\ \\ \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

We get the following equivalent portfolio

$$(\theta_0, \theta_1, \theta_2) = (1, 2, 2)$$

and then, the security X is attainable.

3) $\mathcal{Q} = (q_1, q_2, q_3, q_4)$ is a risk-neutral probability distribution if
— $\forall j \in [\![1, 4]\!], \quad q_j > 0,$
— $q_1 + q_2 + q_3 + q_4 = 1,$

— deflated security prices are martingales. That is,

$$\forall i \in [\![0,2]\!] : \quad \frac{S_0^{(i)}}{1} = \mathbb{E}^{\mathcal{Q}}\left(\frac{S_1^{(i)}}{1+r}\right) = \frac{1}{1+r}\mathbb{E}^{\mathcal{Q}}\left(S_1^{(i)}\right) = \frac{1}{1+r}\sum_{j=1}^{4} q_j S_1^{(i)}(\omega_j)$$

that is equivalent to

$$S_0 = \frac{1}{1+r}\, S_1 \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \Leftrightarrow S_1 \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = 1.05\, S_0.$$

We want solve

$$\begin{array}{cccc|c} 1.05 & 1.05 & 1.05 & 1.05 & 1.05 \\ 4 & 2 & 5 & 3 & 1.05\times 3 \\ 1 & 4 & 2 & 4 & 1.05\times 3 \end{array} \qquad L_1 \leftarrow L_1/1.05$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 5 & 3 & 3.15 \\ 1 & 4 & 2 & 4 & 3.15 \end{array} \qquad \begin{array}{l} L_2 \leftarrow L_2 - 4L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 & -0.85 \\ 0 & 3 & 1 & 3 & 2.15 \end{array} \qquad L_2 \leftarrow L_2 + L_3$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1.3 \\ 0 & 3 & 1 & 3 & 2.15 \end{array} \qquad L_3 \leftarrow L_3 - 3L_2$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1.3 \\ 0 & 0 & -5 & -3 & -1.75 \end{array} \qquad L_3 \leftarrow L_3/(-5)$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1.3 \\ 0 & 0 & 1 & 3/5 & 0.35 \end{array} \qquad L_2 \leftarrow L_2 - 2L_3$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 4/5 & 0.6 \\ 0 & 0 & 1 & 3/5 & 0.35 \end{array} \qquad L_1 \leftarrow L_1 - L_2 - L_3$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & 0 & 0 & -2/5 & 0.05 \\ 0 & 1 & 0 & 4/5 & 0.6 \\ 0 & 0 & 1 & 3/5 & 0.35 \end{array} \qquad L_1 \leftarrow L_1 - L_2 - L_3$$

Then, for all $q_4 \in \mathbb{R}$,

$$\left(0.05 + \frac{2}{5}q_4, 0.6 - \frac{4}{5}q_4, 0.35 - \frac{3}{5}q_4, q_4\right)$$

is solution of

$$S_0 = \frac{1}{1+r} \, S_1 \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}.$$

Since, we want $q_1 > 0$, $q_2 > 0$, $q_3 > 0$ and $q_4 > 0$, we need

$$\begin{cases} 0.05 + \frac{2}{5}q_4 > 0 \\ 0.6 - \frac{4}{5}q_4 > 0 \\ 0.35 - \frac{3}{5}q_4 > 0 \\ q_4 > 0 \end{cases} \Leftrightarrow \begin{cases} q_4 > -\frac{5\times 0.05}{2} = 0.125 \\ q_4 < \frac{5\times 0.6}{4} = 0.75 \\ q_4 < \frac{5\times 0.35}{3} \approx 0.583 \\ q_4 > 0 \end{cases}$$

Then, for all $q_4 \in]0.125, 0.583[$,

$$\left(0.05 + \frac{2}{5}q_4, 0.6 - \frac{4}{5}q_4, 0.35 - \frac{3}{5}q_4, q_4\right)$$

is a risk-neutral probability distribution. For example, with $q_4 = 0.2$, we obtain that

$$\mathcal{Q} = (0.13, 0.44, 0.23, 0.20)$$

is a risk-neutral probability distribution.

4) Since there exists a risk-neutral probability distribution, the fundamental theorem of asset pricing (part 1) gives us that there is no possible arbitrage. Then, there is no arbitrage of type A and then, by a proposition of the course, the linear pricing hypothesis is satisfied.

5) By the fundamental theorem of asset pricing, we know that there exists a vector of states prices. Moreover, since the linear pricing hypothesis is satisfied, $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ is a vector of state prices if

$$\begin{cases} (\pi_1, \pi_2, \pi_3, \pi_4) > 0, \\ S_1 \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = S_0. \end{cases}$$

We saw in question 3) that a risk-neutral probability distribution if and only if

$$S_0 = \frac{1}{1+r} S_1 \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} > 0.$$

Then, we obtain

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4) \text{ is a vector of state prices}$$
$$\Leftrightarrow \frac{\pi}{1+r} = \left(\frac{\pi_1}{1+1}, \frac{\pi_2}{1+r}, \frac{\pi_3}{1+r}, \frac{\pi_4}{1+r}\right) \text{ is a risk-neutral probability distribution.}$$

Then, for all $q_4 \in]0.125, 0.583[$,

$$(1+r) \times \left(0.05 + \frac{2}{5}q_4, 0.6 - \frac{4}{5}q_4, 0.35 - \frac{3}{5}q_4, q_4\right)$$

is a a vector of state prices. For example, with $q_4 = 0.2$, we obtain that

$$\pi = (0.1365, 0.4620, 0.2415, 0.2100)$$

is a vector of state prices.

6) Since linear pricing hypothesis holds, the fair price at time $t = 0$ associated to portfolio $\theta = (1, 2, 2)$ is

$$P = \theta S_0 = (1, 2, 2) \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$
$$= 1 + 2 \times 3 + 2 \times 3 = 1 + 6 + 6 = 13.$$

Then, it is the fair price.

Problem 2 :

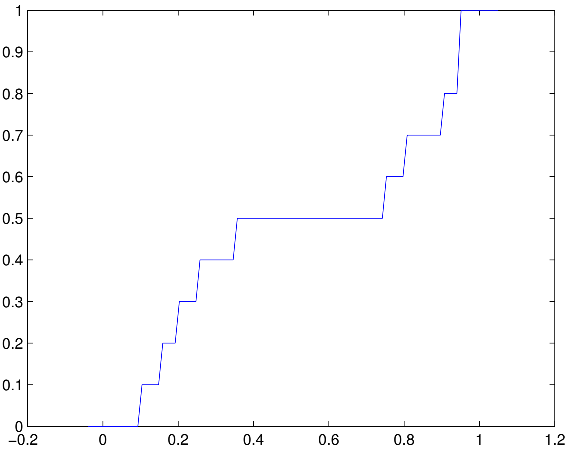
1) The cumulative function is defined by

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

where f is the density function. If we have a random generator $(y_i)_{1\leq i\leq N}$. We can estimate the cumulative function associate to $(y_i)_{1\leq i\leq N}$ with

$$F_N(x) = \frac{\#\{i : y_i \leq x\}}{N}.$$

2) We obtain



3) The density function for a uniform law is

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, the cumulative function for a uniform law is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} \int_{-\infty}^{x} 0dt & \text{if } x \leq 0, \\ \int_{-\infty}^{0} 0dt + \int_{0}^{x} 1dt & \text{if } 0 \leq x \leq 1, \\ \int_{-\infty}^{0} 0dt + \int_{0}^{1} 1dt + \int_{1}^{x} 0dt & \text{elsewhere.} \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{elsewhere.} \end{cases}$$

4) This is not a uniform distribution since the cumulative function is far from the line of equation $y = x$.

Problem 3 :

1) We have $y_0 = 107$, $a = 327$, $c = 1$, $m = 1000$ and linear congruencies sequence is defined by

$$y_{n+1} = a \times y_n + c \mod m.$$

Then, we obtain

$$\begin{aligned} y_1 &= a \times y_n + c \mod m \\ &= 327 \times 107 + 1 \mod 1000 = 34990 \mod 1000 \\ &= 990. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} y_2 &= 327 \times 990 + 1 \mod 1000 \\ &= 323731 \mod 1000 = 731 \\ y_3 &= 327 \times 731 + 1 \mod 1000 \\ &= 239038 \mod 1000 = 38 \\ y_4 &= 327 \times 38 + 1 \mod 1000 \\ &= 12427 \mod 1000 = 427. \end{aligned}$$

2) We just have to divide by $m = 1000$, we obtain

$$z_0 = 0.107, \quad z_1 = 0.99, \quad z_2 = 0.731, \quad z_3 = 0.038, \quad z_4 = 0.427.$$

3) f is a density function because f is positive and

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^{3^{1/3}} x^2 dx = \left[\frac{x^3}{3}\right]_0^{3^{1/3}} = \frac{1}{3}\left(3^{1/3}\right)^3 = \frac{1}{3}3 = 1.$$

4) The cumulative function F of the random variable X is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} \int_{-\infty}^{x} 0dt & \text{if } x \leq 0, \\ \int_{-\infty}^{0} 0dt + \int_0^x t^2 dt & \text{if } 0 \leq x \leq 3^{1/3}, \\ \int_{-\infty}^{0} 0dt + \int_0^{3^{1/3}} t^2 dt + \int_{3^{1/3}}^{x} 0dt & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 0 & \text{if } \leq 0, \\ \left[\frac{x^3}{3}\right]_0^x & \text{if } 0 \leq x \leq 3^{1/3}, \\ \left[\frac{x^3}{3}\right]_0^{3^{1/3}} & \text{elsewhere} \end{cases} = \begin{cases} 0 & \text{if } \leq 0, \\ \frac{x^3}{3} & \text{if } 0 \leq x \leq 3^{1/3}, \\ 1 & \text{elsewhere.} \end{cases}$$

5) We know that if Z is a uniform random variable on $[0, 1]$, the random variable $X = F^{-1}(Z)$ has density f. Then,

$$F^{-1}(z_1), \quad F^{-1}(z_2), \quad F^{-1}(z_3)$$

are three realizations of random variable X. We just have to prove that F is bijective and to compute F^{-1}. With the question 4), we obtain that F is a bijective function from $\left[0, 3^{1/3}\right]$ to $[0, 1]$. We need to compute $F^{-1}(y)$ for some $y \in [0, 1]$, we have

$$\begin{aligned} y = F(x) \Leftrightarrow y = \frac{x^3}{3} \Leftrightarrow x^3 = 3y \Leftrightarrow x = (3y)^{1/3} \\ \Leftrightarrow x = (3y)^{1/3} = F^{-1}(y). \end{aligned}$$

Then,

$$(3z_1)^{1/3} = 1.44, \quad (3z_2)^{1/3} = 1.299, \quad (3z_3)^{1/3} = 0.485$$

are three realizations of random variable X.

6) We have

```
function z= my_generator(a,c,m,y0,k)
    y=congruencial(a,c,m,y0,k);
    for i=1:1:k
        z(i)=(3*y(i))^(1/3)
    end
end
```

7) We know that if Z_1 and Z_2 are two uniform random variables on $[0,1]$, the random variable

$$\sigma\,\sqrt{-2\,\ln(Z_1)}\,cos(2\,\pi\,Z_2)+\mu,$$

is a normal random variable with mean μ and standard deviation σ. In our case, we want a standard normal distribution, then we use $\mu = 0$ and $\sigma = 1$ in the Box and Miller formula. Then,

$$\sqrt{-2\,\ln(z_1)}\,cos(2\,\pi\,z_2) = -0.0169,$$
$$\sqrt{-2\,\ln(z_3)}\,cos(2\,\pi\,z_4) = -2.2931$$

are two realizations of a standard normal random variable.

Problem 4 :
We assume that the following data are values of the price of an security at time $T = 100$ obtained by performing a Monte-Carlo simulations :

$$125,\ 135,\ 95,\ 100.$$

Assume that this security follows a binomial model with $T = 100$.

1) The four simulations give us the following payoffs :

$$15,\ 25,\ 0,\ 0.$$

The average of these four values is

$$\theta_4 = \frac{15+25+0+0}{4} = \frac{40}{4} = 10.$$

The value of the call obtained with the Monte-Carlo method is

$$C = \frac{\theta_4}{(1+r)^{100}} = \frac{10}{(1+0.02)^{100}} \approx 1.38.$$

2) To give a confidence interval, we firstly need to compute the standard devia-

tion s of the four previous value, we have

$$\begin{aligned} s &= \sqrt{\frac{(15-10)^2+(25-10)^2+(0-10)^2+(0-10)^2}{4}} \\ &= \sqrt{\frac{5^2+15^2+10^2+10^2}{4}} \approx 10.60. \end{aligned}$$

Since the confidence level is 95%, we have $Z = 1.96$ and the confidence interval is $[l, u]$ with

$$\begin{aligned} l &= \theta_4 - Z\frac{s}{\sqrt{n}} = 10 - 1.96 \times \frac{10.60}{\sqrt{4}} \approx -0.39 \\ u &= \theta_4 + Z\frac{s}{\sqrt{n}} = 10 + 1.96 \times \frac{10.60}{\sqrt{4}} \approx 20.39, \end{aligned}$$

because

$$\begin{aligned} &P\left(-Z \leq \frac{\theta_n - \theta}{s/\sqrt{n}} \leq Z\right) = 95\% = 0.95 \\ \Leftrightarrow &P\left(\theta_n - Z \times \frac{s}{\sqrt{n}} \leq \theta \leq \theta_n + Z \times \frac{s}{\sqrt{n}}\right) = 0.95 \end{aligned}$$

and it is assumed with the Central Limit Theorem that $\frac{\theta_n-\theta}{s/\sqrt{n}}$ is a standard normal distribution.

We assume that the value obtained in Question 1) are used as the first step of a two phases methods for performing a confidence interval.

3) We have

$$\begin{aligned} &P\left(-Z \leq \frac{\theta_n - \theta}{s\sqrt{n}} \leq Z\right) = 95\% = 0.95 \\ \Leftrightarrow &P\left(-Z \times \frac{s}{\sqrt{n}} \leq \theta_n - \theta \leq Z \times \frac{s}{\sqrt{n}}\right) = 0.95 \\ \Leftrightarrow &P\left(|\theta_n - \theta| \leq Z \times \frac{s}{\sqrt{n}}\right) = 0.95. \end{aligned}$$

Then, the absolute error $|\theta_n - \theta|$ is less than $\epsilon = 3$ if

$$Z \times \frac{s}{\sqrt{n}} \leq \epsilon \Leftrightarrow n \geq \frac{s^2 \times Z^2}{\epsilon^2} \approx \frac{10.60^2 \times 1.96^2}{3^2} = 47.96.$$

Then, to obtained an absolute error of at most 3 with a confidence level of 95%, we need at least

$$n = 48$$

simulations.

4) We have

$$
\begin{aligned}
&P\left(-Z \leq \frac{\theta_n - \theta}{s/\sqrt{n}} \leq Z\right) = 95\% = 0.95 \\
\Leftrightarrow &P\left(-Z \times \frac{s}{\sqrt{n}} \leq \theta_n - \theta \leq Z \times \frac{s}{\sqrt{n}}\right) = 0.95 \\
\Leftrightarrow &P\left(|\theta_n - \theta| \leq Z \times \frac{s}{\sqrt{n}}\right) = 0.95 \\
\Leftrightarrow &P\left(\left|\frac{\theta_n - \theta}{\theta}\right| \leq \frac{Z \times s}{|\theta|\sqrt{n}}\right) = 0.95.
\end{aligned}
$$

Then, the relative error $\left|\frac{\theta_n-\theta}{\theta}\right|$ is less than $\epsilon = 3\% = 0.03$ if

$$
\frac{Z \times s}{|\theta_4|\sqrt{n}} \leq \epsilon \Leftrightarrow n \geq \frac{s^2 \times Z^2}{\theta_4^2 \times \epsilon^2} \approx \frac{10.60^2 \times 1.96^2}{10^2 \times 0.03^2} = 4796.02.
$$

Then, to obtained an relative error of at most $3\% = 0.03$ with a confidence level of 95%, we need at least

$$
n = 4797
$$

simulations.