Advanced Algorithms for Finance
Master of Financial Engineering - M2
Final Exam 2014/2015 - Duration : 2h
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No documents - <u>You can use a calculator</u>

Please report clear and detailed answers explicitly, as well as all the formulas you use and all the theorems you invoke.

Problem 1 : We consider a model over one period of three securities indexed by $i = 0, 1, 2$. This model admits four states at time $t = 1$, we note these states ω_1, ω_2, ω_3 and ω_4. The price of the securities are given by the following vectors

- At time $t = 0$, $S_0 = (1, 3, 3)^T$,
- At time $t = 1$ in state ω_1, $S_1(\omega_1) = (1.05, 4, 1)^T$,
- At time $t = 1$ in state ω_2, $S_1(\omega_2) = (1.05, 2, 4)^T$,
- At time $t = 1$ in state ω_3, $S_1(\omega_3) = (1.05, 5, 2)^T$,
- At time $t = 1$ in state ω_4, $S_1(\omega_4) = (1.05, 3, 4)^T$.

1) In this model, is there a numeraire security ? If it is the case, what is the risk-free rate r ?
2) Somebody sells a security whose price at time $t = 1$ is given by :
   - 11.05 € if ω_1 occurs,
   - 13.05 € if ω_2 occurs,
   - 15.05 € if ω_3 occurs,
   - 15.05 € if ω_4 occurs.

   Is this security attainable ? If it is attainable, please provide an equivalent port-folio. Report all your computations explicitly.
3) Is there a risk-neutral probability distribution in this model ? If there is one, compute such distribution explicitly.
4) Is the linear pricing hypothesis satisfied ?
5) Is there a vector of state prices π ? If there is one, deduce its expression from question 3).
6) The person selling the security described in question 2) sells it for the price of 13 € at time $t = 0$. Is this price fair ?

Problem 2 :

1) Give a formula to *estimate* the cumulative function of a random generator $(y_i)_{1\le i\le N}$.

Assume that the following 10 values are random numbers generated using a congruential generator :

$$0.15, 0.80, 0.90, 0.25, 0.10, 0.95, 0.35, 0.95, 0.75, 0.20.$$

2) Draw the *estimation* of this generator using only these ten values.
3) What is the density function for a uniform law on $[0,1]$? Compute the cumulative function of this uniform law ?
4) Based on the estimation drawn in question 1), would you say this generator is uniform ? Why ?

Problem 3 : Consider a linear congruential generator with parameters a, c, m, and y_0, that produces a sequence (y_n) of numbers in $\mathbb{N}$.

1) Assume that $y_0 = 107$, $a = 327$, $c = 1$, and $m = 1000$. Using these values compute y_n for $0 \le n \le 4$.
2) Give the realizations z_n for $0 \le n \le 4$ of the distributed sequence in $[0,1]$ associated to (y_n).
3) Consider the function f defined over $\mathbb{R}$ by :

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 3^{1/3}, \\ 0 & \text{otherwise.} \end{cases}$$

   Explain why f is the density function of some random variable X (please support your explanation with a calculation).
4) Compute the cumulative function of random variable X.
5) Provide three realizations of random variable X obtained by the inverse transform method with the values of z_1, z_2, and z_3 found at question 2).
6) Assume that you have a congruential random number generator written in Matlab called *congruential*. You call this generator as follows :

$$\text{congruential}(a, c, m, y0, k)$$

   This command returns a column vector containing the first k terms of the pseudo-random sequence generated by the congruential number generator with parameters a, c, m, and seed $y0$ (the terms of the sequence are in $[0,1[$). Write a Matlab function my_generator that takes arguments a,c, m, $y0$, and k and that returns k realizations of random variable X, generated using the inverse transform method. Include the function header in your code.
7) Using Box and Miller method, provide two realizations of a standard normal

random variable with the values of z_1, z_2, z_3 and z_4 found at question 2).

Problem 4 :
We assume that the following data are values of the price of an security at time $T = 100$ obtained by performing a Monte-Carlo simulations :

$$125,\ 135,\ 95,\ 100.$$

Assume that this security follows a binomial model with $T = 100$.

1) With the Monte-Carlo method, give an estimation of the price of a european call with strike $K = 110$ at expiration date $T = 100$. We assume hier that the risk-free rate r is independent of the time and is equal to 0.02.
2) Give a confidence interval at level 95% for the estimation found in Question 1).

We assume that the value obtained in Question 1) are used as the first step of a two phases methods to build a confidence interval.

3) How many simulations must be carried out in the second step to obtain an absolute error of at most 3 with a confidence level of 95% ?
4) How many simulations must be carried out in the second step to obtain an relative error of at most 3% with a confidence level of 95% ?