Black and Scholes model
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1 Brownian motion

Definition 1.1. ***(Stochastic process)*** *A stochastic process S_t is set of random variables indexed to the time t.*

Remark 1.2. *In discrete pricing models, the price S_t of a security is a stochastic process because the price S_t depends on the state of the world that happens at time t. Then, at each time t, S_t is a random variable.*

Definition 1.3. ***(Brownian motion)*** *A stochastic process B_t is a standard Brownian motion, or Wiener process, if*

1. $B_0 = 0$,
2. *Non-overlapping increments are independent. It means that $\forall 0 \leq s < S \leq t < T$, the increments $B_T - B_t$ and $B_S - B_s$ are independent random variables,*
3. *$\forall 0 \leq s < t$, the increment $B_t - B_s$ is a normal random variable, with zero mean and variance $t - s$,*
4. *for all ω, the path $t \mapsto S_t(\omega)$ is continuous.*

Then, for each $t > 0$, the random variable $B_t = B_t - B_0$ is the increment in $[0, t]$: it is normally distributed with zero mean, variance t and density

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Definition 1.4. ***(Geometric Brownian motion)*** *A stochastic process S_t is a geometric Brownian motion with parameters μ and σ if*

$$S_t = S_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t}$$

where B_t is the time t value of a standard Brownian motion.

Proposition 1.5. *If S_t is a geometric Brownian motion with parameters μ and σ, we have*

$$\forall t, s \geq 0, \quad S_{t+s} = S_t \, e^{\left(\mu - \frac{\sigma^2}{2}\right) s + \sigma(B_{t+s} - B_t)}.$$

Proof. We have

$$\begin{aligned} \forall t, s \geq 0, \quad S_{t+s} &= S_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right) (t+s) + \sigma B_{t+s}} \\ &= S_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t} e^{\left(\mu - \frac{\sigma^2}{2}\right) s + \sigma(B_{t+s} - B_t)} \\ &= S_t \, e^{\left(\mu - \frac{\sigma^2}{2}\right) s + \sigma(B_{t+s} - B_t)}. \end{aligned}$$

□

Corollary 1.6. *If S_t is a geometric Brownian motion with parameters μ and σ. We have*

$$\forall t, s \geq 0, \quad \mathbb{E}_t(S_{t+s}) = S_t \, e^{\mu s}.$$

Proof. Let $t, s \geq 0$, since $S_{t+s} = S_t \, e^{\left(\mu - \frac{\sigma^2}{2}\right) s + \sigma(B_{t+s} - B_t)}$, we have

$$\begin{aligned} \mathbb{E}_t(S_{t+s}) &= \mathbb{E}_t \left(S_t \, e^{\left(\mu - \frac{\sigma^2}{2}\right) s + \sigma(B_{t+s} - B_t)} \right) \\ &= S_t \, e^{\left(\mu - \frac{\sigma^2}{2}\right) s} \, \mathbb{E}_t \left(e^{\sigma(B_{t+s} - B_t)} \right). \end{aligned}$$

Since B_t is a standard Brownian motion, $B_{t+s} - B_t$ is a normal random variable, with zero mean and variance s, and then

$$\begin{aligned}
\mathbb{E}_t(S_{t+s}) &= S_t\, e^{\left(\mu - \frac{\sigma^2}{2}\right)\, s}\, \mathbb{E}_t\left(e^{\sigma(B_{t+s}-B_t)}\right) \\
&= S_t\, e^{\left(\mu - \frac{\sigma^2}{2}\right)\, s} \int_{-\infty}^{+\infty} e^{\sigma x} \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} dx = S_t\, e^{\mu\, s} \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{+\infty} e^{-\frac{\sigma^2}{2}s + \sigma x - \frac{x^2}{2s}} dx \\
&= S_t\, e^{\mu\, s} \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{+\infty} e^{-\frac{\left(\frac{x}{\sqrt{s}} + \sigma\sqrt{s}\right)^2}{2}} dx.
\end{aligned}$$

By setting, $y = \frac{x}{\sqrt{s}} + \sigma\sqrt{s}$, we obtain

$$\mathbb{E}_t(S_{t+s}) = S_t\, e^{\mu\, s} \frac{\sqrt{s}}{\sqrt{2\pi s}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy = S_t\, e^{\mu\, s}.$$

□

Corollary 1.7. *If S_t is a geometric Brownian motion with parameters μ and σ, the gross return $R_{t,t+s}$ in any period $[t, t+s]$, is independent of returns in earlier periods. In particular, it is independent of S_t. This following by noting*

$$\forall t, s \geq 0, \quad R_{t,t+s} = \frac{S_{t+s}}{S_t} = e^{\left(\mu - \frac{\sigma^2}{2}\right)\, s + \sigma(B_{t+s}-B_t)}.$$

and recalling the independent increments property of the Brownian motion B_t ($B_{t+s} - B_t$ is independent from $B_t = B_t - B_0$).

Proof. We recall that, if X and Y are two independent random variables and if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two continuous functions, $f(X)$ and $g(Y)$ are two independent random variables.
By setting

$$f : x \mapsto e^{\left(\mu - \frac{\sigma^2}{2}\right)\, s + \sigma x} \quad \text{and} \quad g :\mapsto S_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)\, t + \sigma x},$$

f and g are two continuous functions. Since, B_t is a Brownian motion, for all $t, s \geq 0$, $B_{t+s} - B_t$ and $B_t = B_t - B_0$ are independent random variables. Then, for all $t, s \geq 0$,

$$f\left(B_{t+s} - B_t\right) = e^{\left(\mu - \frac{\sigma^2}{2}\right)\, s + \sigma(B_{t+s}-B_t)} \quad \text{and} \quad g\left(B_t\right) = S_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)\, t + \sigma B_t}$$

are independent random variables, meaning that $R_{t,t+s}$ and S_t are independent random variables. □

Remark 1.8. *In finance, we often model the value of a risky security like a geometric Brownian motion. In particular, it is an hypothesis of the Black and Scholes model. In this case, σ is the volubility of this security (that is the standard deviation of returns) and μ is the drift, or the trend of the security (that is the mean variation per time units).*

Remark 1.9. *The binomial model has similar properties since the gross return in any period of the binomial model is either u or d, and this is independent of what has happened in earlier periods.*

2 Black-Scholes

The Black-Scholes formula gives the price of a european call C or put P with an underlying geometric Brownian motion model S_t

$$C = \mathbb{E}^{\mathcal{Q}}\left(\max(S_T - K, 0)\right) = S_0\, \phi(d_1) - Ke^{-r\,T}\phi(d_2) \quad \text{for a call,} \tag{1}$$

$$P = \mathbb{E}^{\mathcal{Q}}\left(\max(K - S_T, 0)\right) = -S_0\, \phi(-d_1) + Ke^{-r\,T}\phi(-d_2) \quad \text{for a put} \tag{2}$$

where $\mathcal{Q}$ is the risk-neutral probability distribution, T is the exercice date, K is the strike and ϕ est the cumulative of the standard normal distribution:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{w^2}{2}} dw. \tag{3}$$

and

$$\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \\
d_2 &= d_1 - \sigma\sqrt{T}.
\end{aligned}$$

Remark 2.1. *This formula appears, in the Black and Scholes model, by using the stochastic computation.*

We note that ϕ can also be written as

$$\phi(z) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right),$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-w^2}dw.$$

3 Calibrating the binomial model and convergence to Black-Scholes

3.1 Calibrating the binomial model: the model of Cox Ross-Rubinstein

We often wish to calibrate the binomial model M_k so that its dynamics match that of the geometric Brownian motion S_t with expiration date T. Note that for the geometric Brownian motion we have $0 \le t \le T$ and for the binomial model we have $0 \le k \le n$ where n corresponds to the number of period, then we have $\Delta t = \frac{T}{n}$. To do this we need to choose u, d, q_1 and $q_2 = 1 - q_1$ of the discret model such that its correspond to parameters μ and σ of the geometric Brownian motion. There are many possible ways of doing this, but one of the more common choices is to set

$$p_n = \frac{e^{\mu\frac{T}{n}} - d_n}{u_n - d_n}$$

$$u_n = e^{\sigma\sqrt{\frac{T}{n}}} \tag{4}$$

$$d_n = \frac{1}{u_n} = e^{-\sigma\sqrt{\frac{T}{n}}} \tag{5}$$

where p_n is the <u>real</u> probability of an growth and $1 - p_n$ is the <u>real</u> probability of an reduction of the price of the security. With these parameters, we have for all $0 \le k \le n$

$$\begin{aligned}\mathbb{E}_k^{\mathcal{P}}(M_{k+1}) &= \mathbb{E}^{\mathcal{P}}(M_{k+1}|M_k) = p_n u_n M_k + (1-p_n)d_n M_k = p_n(u_n - d_n)M_k + d_n M_k \\ &= e^{\mu\frac{T}{n}}M_k - d_n M_k + d_n M_k = e^{\mu\frac{T}{n}}M_k\end{aligned}$$

as desired. Indeed, with Corollary 1.6, we have

$$\forall t \ge 0, \quad \mathbb{E}_t\left(S_{t+\frac{T}{n}}\right) = S_t\, e^{\mu\frac{T}{n}}.$$

We will choose the growth risk-free rate per period $1 + r_n$ so that it corresponds to a continuously compounded rate r in continuous time. We therefore have

$$r_n = e^{r\frac{T}{n}} - 1. \tag{6}$$

Remark 3.1. *Recall that the true probability of an up-move, p, has no bearing upon the risk-neutral probability q, and therefore it does not directly affect how securities are priced. On the other hand, u and d depend on σ which therefore does impact security prices. This is a recurring theme in derivatives pricing and we will revisit it when we study continuous time models.*

Exercice : What is the condition on n to have a risk-neutral probability distribution $(q_n, 1 - q_n)$? In this case, give the expression of q_n.

Solution: There exists a risk-neutral probability distribution if

$$\begin{aligned}d_n < 1 + r_n < u_n \Leftrightarrow e^{-\sigma\sqrt{\frac{T}{n}}} < e^{\frac{rT}{n}} < e^{\sigma\sqrt{\frac{T}{n}}} &\Leftrightarrow -\sigma\sqrt{\frac{T}{n}} < \frac{rT}{n} < \sigma\sqrt{\frac{T}{n}} \\ &\Leftrightarrow \sqrt{n} > \frac{r\sqrt{T}}{\sigma} \Leftrightarrow n > \frac{r^2T}{\sigma^2}.\end{aligned} \tag{7}$$

If (7) is satisfied, the risk-neutral probability distribution $(q_n, 1 - q_n)$ is given by

$$q_n = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}}. \tag{8}$$

In the sequel, we assume that n satisfies (7) such that $(q_n, 1 - q_n)$ is a risk-neutral probability distribution.

3.2 Convergence of the binomial model to Black-Scholes

For a fixed expiration date T, we consider the sequence of binomial models M_n that are parametrized by (4)-(5) and (6). We now explain how (1) may be obtained by setting $n \to \infty$ in this sequence.

Proposition 3.2. *Assume that there is no possible arbitrage.*
The price C of a European call of a security following a binomial model M_k (with parameters u, d and r) with initial state M_0, expiration date at n periods and strike K is given by

$$C = M_0 \, D\left(n, \eta_n, \frac{q\,u}{1+r}\right) - \frac{K}{(1+r)^n} \, D\left(n, \eta_n, q\right) \tag{9}$$

where $q = \frac{1+r-d}{u-d}$ and

$$\eta_n = int\left(\frac{\ln\left(\frac{K}{M_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)}\right) + 1 \tag{10}$$

$$D(n,l,q) = \sum_{j=l}^{n} \binom{n}{j} q^j (1-q)^{n-j}. \tag{11}$$

Proof. Since the model is without arbitrage, we have $q = \frac{1+r-d}{u-d}$. Since the model is binomial, there is $n+1$ possible states $\omega_0, ..., \omega_n$ at time n. The value of this security at time n is

$$\forall j \in [\![0,n]\!], \quad M_n(\omega_j) = u^j \, d^{n-j} M_0,$$

and the risk-neutral probability distribution $\mathcal{Q} = (q_0, ..., q_n)$ satisfies

$$\forall j \in [\![0,n]\!], \quad q_j = \binom{n}{j} q^j (1-q)^{n-j}.$$

Then,

$$\begin{aligned} C = \frac{\mathbb{E}_0^{\mathcal{Q}}\big(\max(M_n - K, 0)\big)}{(1+r)^n} &= \frac{1}{(1+r)^n} \sum_{j=0}^{n} q_j \, \max\big(M_n(\omega_j) - K, 0\big) \\ &= \frac{1}{(1+r)^n} \sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} \, \max\big(u^j d^{n-j} M_0 - K, 0\big). \end{aligned}$$

However, since $u > d$, we have

$$\begin{aligned} u^j d^{n-j} M_0 - K > 0 \Leftrightarrow \left(\frac{u}{d}\right)^j > \frac{K}{M_0 d^n} &\Leftrightarrow j \ln\left(\frac{u}{d}\right) > \ln\left(\frac{K}{M_0 d^n}\right) \\ \Leftrightarrow j > \frac{\ln\left(\frac{K}{M_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)} &\Leftrightarrow j \geq \eta_n \end{aligned}$$

where η_n is defined by (10). Then,

$$\begin{aligned} C =& \frac{1}{(1+r)^n} \sum_{j=\eta_n}^{n} \binom{n}{j} q^j (1-q)^{n-j} \left(u^j d^{n-j} M_0 - K\right) \\ =& \frac{M_0}{(1+r)^n} \sum_{j=\eta_n}^{n} \binom{n}{j} (u\,q)^j \left((1-q)\,d\right)^{n-j} \\ &- \frac{K}{(1+r)^n} \sum_{j=\eta_n}^{n} \binom{n}{j} q^j (1-q)^{n-j} \\ =& M_0 \sum_{j=\eta_n}^{n} \binom{n}{j} \left(\frac{u\,q}{1+r}\right)^j \left(\frac{(1-q)\,d}{1+r}\right)^{n-j} \\ &- \frac{K}{(1+r)^n} \sum_{j=\eta_n}^{n} \binom{n}{j} q^j (1-q)^{n-j}. \end{aligned}$$

Since $q = \frac{1+r-d}{u-d}$, we have $\frac{(1-q)d}{1+r} = 1 - \frac{uq}{1+r}$ and then we obtain (9). □

Then, if we assume that the binomial parameters are given by (4)-(5), (6) and (8), the price C_n of a European call of a security following a binomial model M_n (with parameters u_n, d_n and r_n) with initial state M_0, expiration date at n periods and strike K is given by

$$C_n = M_0\, D\left(n, \eta_n, \frac{q_n\, u_n}{1+r_n}\right) - \frac{K}{(1+r_n)^n}\, D\left(n, \eta_n, q_n\right). \tag{12}$$

Note the similarity between the Black-Scholes formula (1) and (12). Clearly all that is now required is to show that

$$D\left(n, \eta_n, \frac{q_n\, u_n}{1+r_n}\right) \underset{n\to\infty}{\longrightarrow} \phi\left(\frac{\ln\left(\frac{M_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$D\left(n, \eta_n, q_n\right) \underset{n\to\infty}{\longrightarrow} \phi\left(\frac{\ln\left(\frac{M_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right)$$

where $D(n, l, q)$ and $\phi(d)$ are respectively defined by (11) and (3).

Lemma 3.3.

$$D(n, \eta_n, q_n) \underset{n\to\infty}{\longrightarrow} \phi\left(\frac{\ln\left(\frac{M_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right)$$

Proof. We have

$$\begin{aligned} D(n, \eta_n, q_n) &= \sum_{j=\eta_n}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} = 1 - \sum_{j=0}^{\eta_n - 1} \binom{n}{j} q_n^j (1-q_n)^{n-j} \\ &= 1 - P(X_n < \eta_n) \end{aligned}$$

where X_n is the sum of n independent Bernouilli random variables with parameters q_n. We can therefore write

$$D(n, \eta_n, q_n) = 1 - P(X_n < \eta_n) = 1 - P\left(\frac{X_n - nq_n}{\sqrt{nq_n(1-q_n)}} < \frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}}\right).$$

By applying the Central Limit Theorem, we have

$$P\left(\frac{X_n - nq_n}{\sqrt{nq_n(1-q_n)}} < \frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}}\right) \underset{n\to\infty}{\longrightarrow} \phi\left(\lim_{n\to\infty} \frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}}\right) \tag{13}$$

where we recall that ϕ is the cumulative function of the standard normal distribution. Then, we need to compute $\lim\limits_{n\to\infty} \frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}}$. We have

$$\eta_n - nq_n = int\left(\sqrt{n}\frac{\ln\left(\frac{K}{M_0}\right)}{2\sigma\sqrt{T}} + \frac{n}{2}\right) + 1 - n\frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}}$$

where

$$\begin{aligned} nq_n &= n\frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}} \\ &= \frac{\sigma\sqrt{nT} + \left(r - \frac{\sigma^2}{2}\right)T + \frac{\sigma^3 T^{3/2}}{6\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)}{\frac{2\sigma\sqrt{T}}{\sqrt{n}} + \frac{\sigma^3 T^{3/2}}{3n^{3/2}} + o\left(\frac{1}{n^2}\right)} \\ &= \frac{\sigma\sqrt{nT} + \left(r - \frac{\sigma^2}{2}\right)T + \frac{\sigma^3 T^{3/2}}{6\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)}{\frac{2\sigma\sqrt{T}}{\sqrt{n}}\left(1 + \frac{\sigma^2 T}{6n} + o\left(\frac{1}{n^{3/2}}\right)\right)} \\ &= \frac{\frac{1}{2}n + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{T}\sqrt{n} + \frac{\sigma^2 T}{12} + o(1)}{1 + \frac{\sigma^2 T}{6n} + o\left(\frac{1}{n^{3/2}}\right)} \\ &= \left(\frac{1}{2}n + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{T}\sqrt{n} + \frac{\sigma^2 T}{12} + o(1)\right)\left(1 - \frac{\sigma^2 T}{6n} - o\left(\frac{1}{n^{3/2}}\right)\right) \\ &= \frac{1}{2}n + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{T}\sqrt{n} + o(1) \end{aligned}$$

and

$$\sqrt{n}\frac{\ln\left(\frac{K}{M_0}\right)}{2\sigma\sqrt{T}}+\frac{n}{2}\leq\eta_n=int\left(\sqrt{n}\frac{\ln\left(\frac{K}{M_0}\right)}{2\sigma\sqrt{T}}+\frac{n}{2}\right)+1<\sqrt{n}\frac{\ln\left(\frac{K}{M_0}\right)}{2\sigma\sqrt{T}}+\frac{n}{2}+1.$$

Moreover, we have

$$\begin{aligned}
1-q_n &= \frac{e^{\sigma\sqrt{\frac{T}{n}}}-e^{r\frac{T}{n}}}{e^{\sigma\sqrt{\frac{T}{n}}}-e^{-\sigma\sqrt{\frac{T}{n}}}}\\
&= \frac{\sigma\sqrt{\frac{T}{n}}+\left(\frac{\sigma^2}{2}-r\right)\frac{T}{n}+o\left(\frac{1}{n}\right)}{\frac{2\sigma\sqrt{T}}{\sqrt{n}}+o\left(\frac{1}{n}\right)}\\
&= \frac{\sigma\sqrt{\frac{T}{n}}+\left(\frac{\sigma^2}{2}-r\right)\frac{T}{n}+o\left(\frac{1}{n}\right)}{\frac{2\sigma\sqrt{T}}{\sqrt{n}}\left(1+o\left(\frac{1}{\sqrt{n}}\right)\right)}\\
&= \frac{\frac{1}{2}+\frac{\frac{\sigma^2}{2}-r}{2\sigma}\sqrt{\frac{T}{n}}+o\left(\frac{1}{\sqrt{n}}\right)}{1+o\left(\frac{1}{\sqrt{n}}\right)}\\
&= \left(\frac{1}{2}+\frac{\frac{\sigma^2}{2}-r}{2\sigma}\sqrt{\frac{T}{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right)\left(1-o\left(\frac{1}{\sqrt{n}}\right)\right)\\
&= \frac{1}{2}+\frac{\frac{\sigma^2}{2}-r}{2\sigma}\sqrt{\frac{T}{n}}+o\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

and

$$\begin{aligned}
nq_n(1-q_n) &= \left(\frac{1}{2}n+\frac{r-\frac{\sigma^2}{2}}{2\sigma}\sqrt{T}\sqrt{n}+o(1)\right)\left(\frac{1}{2}+\frac{\frac{\sigma^2}{2}-r}{2\sigma}\sqrt{\frac{T}{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right)\\
&= \frac{n}{4}+o(1)=\frac{n}{4}\left(1+o\left(\frac{1}{n}\right)\right)\\
\Longrightarrow\quad \sqrt{nq_n(1-q_n)} &= \frac{\sqrt{n}}{2}\sqrt{1+o\left(\frac{1}{n}\right)}=\frac{\sqrt{n}}{2}\sqrt{1+o\left(\frac{1}{n}\right)}=\frac{\sqrt{n}}{2}\left(1+\frac{1}{2}o\left(\frac{1}{n}\right)\right)\\
&= \frac{\sqrt{n}}{2}\left(1+o\left(\frac{1}{n}\right)\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\eta_n-nq_n}{\sqrt{nq_n(1-q_n)}} &\leq \frac{\sqrt{n}\frac{\ln\left(\frac{K}{M_0}\right)}{2\sigma\sqrt{T}}+\frac{n}{2}+1-\left(\frac{1}{2}n+\frac{r-\frac{\sigma^2}{2}}{2\sigma}\sqrt{T}\sqrt{n}+o(1)\right)}{\frac{\sqrt{n}}{2}\left(1+o\left(\frac{1}{n}\right)\right)}\\
&= \frac{\frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}}-\frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}+\frac{2}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)}{1+o\left(\frac{1}{n}\right)}\\
&= \left(\frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}}-\frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}+\frac{2}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right)\left(1-o\left(\frac{1}{n}\right)\right)\\
&= \frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}}-\frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}+\frac{2}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\\
&\underset{n\to\infty}{\longrightarrow} \frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}}-\frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}. \qquad (14)
\end{aligned}$$

and similarly

$$
\begin{aligned}
\frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}} &\geq \frac{\sqrt{n}\frac{\ln\left(\frac{K}{M_0}\right)}{2\sigma\sqrt{T}} + \frac{n}{2} - \left(\frac{1}{2}n + \frac{r-\frac{\sigma^2}{2}}{2\sigma}\sqrt{T}\sqrt{n} + o(1)\right)}{\frac{\sqrt{n}}{2}\left(1 + o\left(\frac{1}{n}\right)\right)} \\
&= \frac{\frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} - \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T} + o\left(\frac{1}{\sqrt{n}}\right)}{1 + o\left(\frac{1}{n}\right)} \\
&= \left(\frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} - \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T} + o\left(\frac{1}{\sqrt{n}}\right)\right)\left(1 - o\left(\frac{1}{n}\right)\right) \\
&= \frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} - \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T} + o\left(\frac{1}{\sqrt{n}}\right) \\
&\underset{n\to\infty}{\longrightarrow} \frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} - \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T} \qquad (15)
\end{aligned}
$$

Then, with (14) and (15), we obtain that

$$\frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}} \underset{n\to\infty}{\longrightarrow} \frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} - \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}$$

The equation (13) becomes

$$P\left(\frac{X_n - nq_n}{\sqrt{nq_n(1-q_n)}} < \frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}}\right) \underset{n\to\infty}{\longrightarrow} \phi\left(\frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} - \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}\right)$$

Since,

$$D(n,\eta_n,q_n) = 1 - P(X_n < \eta_n) = 1 - P\left(\frac{X_n - nq_n}{\sqrt{nq_n(1-q_n)}} < \frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}}\right).$$

and since for all $x \in \mathbb{R}$, $\phi(-x) = 1 - \phi(x)$ because

$$\phi(-x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-\frac{u^2}{2}}du = \frac{1}{\sqrt{2\pi}}\int_{x}^{+\infty} e^{-\frac{v^2}{2}}dv = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{v^2}{2}}dv = 1 - \phi(x),$$

we obtain

$$
\begin{aligned}
D(n,\eta_n,q_n) \underset{n\to\infty}{\longrightarrow} 1 - \phi\left(\frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} - \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}\right) &= \phi\left(-\frac{\ln\left(\frac{K}{M_0}\right)}{\sigma\sqrt{T}} + \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}\right) \\
&= \phi\left(\frac{\ln\left(\frac{M_0}{K}\right)}{\sigma\sqrt{T}} + \frac{r-\frac{\sigma^2}{2}}{\sigma}\sqrt{T}\right) \\
&= \phi\left(\frac{\ln\left(\frac{M_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right).
\end{aligned}
$$

□

We can prove in a similar manner that

$$D\left(n, \eta_n, \frac{q_n\, u_n}{1 + r_n}\right) \underset{n\to\infty}{\longrightarrow} \phi\left(\frac{\ln\left(\frac{M_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

and then the price C_n of the call satisfies

$$\begin{aligned} C_n &= M_0\, D\left(n, \eta_n, \frac{q_n\, u_n}{1+r_n}\right) - \frac{K}{(1+r_n)^n}\, D\left(n, \eta_n, q_n\right) \\ &= M_0\, D\left(n, \eta_n, \frac{q_n\, u_n}{1+r_n}\right) - \frac{K}{e^{rT}}\, D\left(n, \eta_n, q_n\right) \\ &\underset{n\to\infty}{\longrightarrow} M_0\, \phi\left(\frac{\ln\left(\frac{M_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right) - K\, e^{-rT}\, \phi\left(\frac{\ln\left(\frac{M_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right) \end{aligned}$$

which is exactly the Black-Scholes formula (1).

Conclusion: For a fixed expiration date T, the call C_n with strike K of the sequence of binomial models M_n that are parametrized by u_n, d_n, r_n and q_n given by (4),(5), (6) and (8) tends to the call C given by the Black-Scholes formula (1) when $n \to \infty$.

Remark 3.4. *With similar argument, we can prove that the put P_n with strike K at expiration date T of a sequence of binomial models M_n that are parametrized by u_n, d_n, r_n and q_n given by* (4),(5), (6) *and* (8) *tends to the put P given b the Black-Scholes formula* (2) *when $n \to \infty$.*