A low Mach correction for the Godunov scheme applied to the linear wave equation with porosity

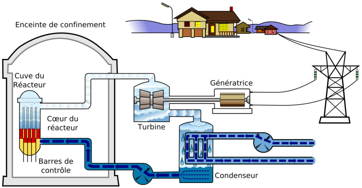
Jonathan Jung (LRC-Manon, Paris 6)

Collaborators :
Stéphane Dellacherie (CEA, Saclay & LRC-Manon)
Pascal Omnes (CEA, Saclay & Université Paris 13)

DSFD, ENS Paris, July 29th 2014

Study case :

- Nuclear reactor core.



Properties of the flow :

- Flow with variable cross section (porosity).
- Liquid-gas flow.
- Compressible flow.
- Low Mach number

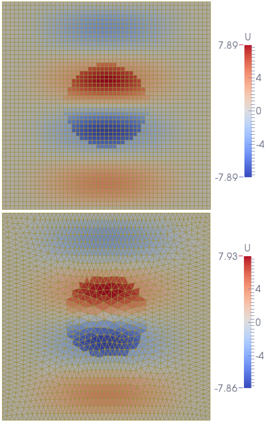
$$|u| \ll c$$
$$\Leftrightarrow M := \frac{|u|}{c} \ll 1.$$

Aim :

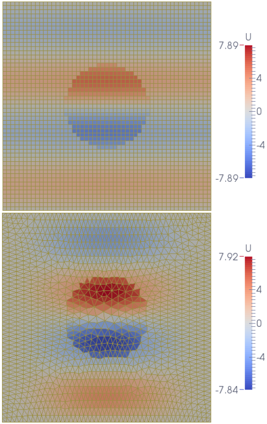
- Develop a "compressible" numerical scheme that is accurate at low Mach number.

Numerical problems : an initial **stationary** condition

- Initial condition :



- After 1000 iterations :



Aims :

- Understand the influence of the cell geometry.
- Propose a correction on cartesian meshes.

Linear wave equations with porosity (or void fraction)

- Dimensionless barotropic Euler equations with porosity (or void fraction)
$$\begin{cases} \partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \frac{\alpha}{M^2}\nabla p = 0, \end{cases}$$
where $\alpha \in [\alpha_{\min}, 1]$ is the porosity, where $\alpha_{\min} > 0$ is a constant that does not depend on M.

- Change of variable $\rho := \rho_\star\left(1 + \frac{M}{a_\star}r\right)$, with $\begin{cases} a_\star^2 = p'(\rho_\star) \\ \frac{M}{a_\star}r \ll 1. \end{cases}$
Linearization around $(\rho_\star = cst, \mathbf{u}_\star = 0)$:
$$\begin{cases} \partial_t(\alpha r) + \frac{a_\star}{M}\nabla\cdot(\alpha\mathbf{u}) = 0, \\ \partial_t(\alpha\mathbf{u}) + \frac{a_\star}{M}\alpha\nabla r = 0. \end{cases}$$

- Kernel of the spatial operator : **incompressible space**
$$\mathcal{E}_\alpha := \left\{ q = (r,\mathbf{u})^T \in L^2_\alpha\left(\mathbb{T}\right)^3 \,\middle|\, \nabla r = 0 \text{ and } \nabla\cdot(\alpha\mathbf{u}) = 0 \right\}.$$

Theorem

We can build a Hodge decomposition on the weighted spaces

$$\mathcal{E}_\alpha \oplus \mathcal{E}_\alpha^\perp = L^2_\alpha\left(\mathbb{T}\right)^3,$$

where the ***acoustic space*** $\mathcal{E}_\alpha^\perp$ *is defined by*

$$\mathcal{E}_\alpha^\perp = \left\{ q = (r, \boldsymbol{u})^T \in L^2_\alpha\left(\mathbb{T}\right)^3 \,\middle|\, \int_{\mathbb{T}} r\alpha dx = 0 \text{ and } \exists \phi \in H^1_\alpha\left(\mathbb{T}\right), \boldsymbol{u} = \nabla\phi \right\}.$$

Proposition

If q is solution of the linear wave equation with porosity (or void fraction) with an initial condition q^0, we have

- $\forall q^0 \in \mathcal{E}_\alpha, \quad q(t \geq 0) \in \mathcal{E}_\alpha;$
- $\forall q^0 \in \mathcal{E}_\alpha^\perp, \quad q(t \geq 0) \in \mathcal{E}_\alpha^\perp.$

Godunov scheme

Godunov scheme on a triangular or a cartesian mesh $(\Omega_i)_{1\leq i\leq N}$

$$\begin{cases} \frac{d}{dt}(\alpha r)_i + \frac{a_\star}{2M}\frac{1}{|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\,(\alpha\mathbf{u}\cdot\mathbf{n})_{ij} = 0, \\ \frac{d}{dt}(\alpha\mathbf{u})_i + \frac{a_\star}{2M}\frac{\alpha_i}{|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|r_{ij}\mathbf{n}_{ij} = 0, \end{cases}$$

where $\left(r_{ij}, (\alpha\mathbf{u}\cdot\mathbf{n})_{ij}\right)$ is the solution of the $1D$ Riemann problem [1] in the $\mathbf{n}_{ij}$ direction

$$\begin{cases} \alpha_{ij}\partial_t r_\xi + \frac{a_\star}{M}\partial_\xi\left((\alpha u)_\xi\right) = 0, \\ \partial_t\left((\alpha u)_\xi\right) + \frac{a_\star}{M}\alpha_{ij}\partial_\xi r_\xi = 0, \\ \left(r_\xi, (\alpha u)_\xi\right)(t=0,\xi) = \begin{cases} \left(r_i, (\alpha\mathbf{u})_i\cdot\mathbf{n}_{ij}\right) & \text{if } \xi<0, \\ \left(r_j, (\alpha\mathbf{u})_j\cdot\mathbf{n}_{ij}\right) & \text{otherwise.} \end{cases} \end{cases}$$

1. S. Dellacherie, P. Omnes, On the Godunov scheme applied to the variable cross-section linear equation. FVCA6, (4) :313–321, 2011.

Numerical scheme

- Godunov scheme on a triangular or a cartesian mesh $(\Omega_i)_{1\leq i\leq N}$ [DO11]

$$\begin{cases} \dfrac{d}{dt}(\alpha r)_i + \dfrac{a_\star}{2M}\dfrac{1}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\Big[\big((\alpha\mathbf{u})_i + (\alpha\mathbf{u})_j\big)\cdot\mathbf{n}_{ij} + \alpha_{ij}(r_i - r_j)\Big] = 0, \\ \dfrac{d}{dt}(\alpha\mathbf{u})_i + \dfrac{a_\star}{2M}\dfrac{\alpha_i}{|\Omega_i|}\displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\Big[r_i + r_j + \dfrac{\kappa}{\alpha_{ij}}\big(\,(\alpha\mathbf{u})_i - (\alpha\mathbf{u})_j\,\big)\cdot\mathbf{n}_{ij}\Big]\mathbf{n}_{ij} = 0 \end{cases}$$

with $\kappa = 1$.

- We write it under the form

$$\frac{d}{dt}(\alpha q_h) + \frac{\mathbb{L}^h_{\kappa,\alpha}}{M}(q_h) = 0,$$

where $q_h = (r_i, \mathbf{u}_i)^T$.

Tools and initial condition

Tools :

- *Mesh :* Software SALOME.
- *Code :* Library C++ CDMATH (in developing in CEA).

Initial condition :

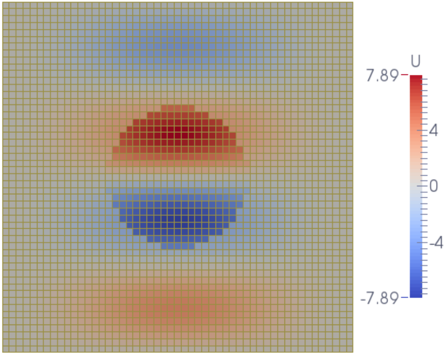
- For all $(x, y) \in [0, 1[\times[0, 1[$,
  - $\alpha(x, y) = \begin{cases} \frac{1}{2} \text{ if } (x, y) \in \mathcal{C}\left(\frac{1}{2}, \frac{1}{2}\right), \\ 1 \text{ otherwise,} \end{cases}$,
  - $r^0(x, y) = 1$,
  - $(\alpha\mathbf{u})^0 = \nabla \times \psi$ where $\psi(x, y) = \frac{2}{\pi}\left(\sin^2(\pi x)\sin^2(2\pi y) - \frac{1}{4}\right)$.

We have

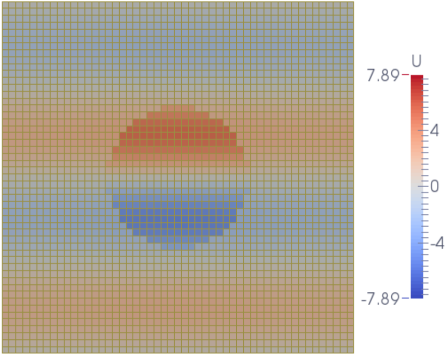
$$\begin{cases} \nabla r^0 = 0 \\ \nabla \cdot (\alpha\mathbf{u}^0) = \nabla \cdot (\nabla \times \psi) = 0 \end{cases} \Rightarrow q^0 = (r^0, \mathbf{u}^0)^T \in \mathcal{E}_\alpha.$$

Cartesian mesh with $\kappa = 1$:

- Initial condition :



- After 1000 iterations :

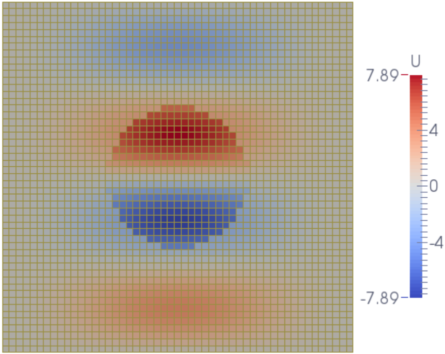


Proposition ($\kappa = 1$ on $\square$)

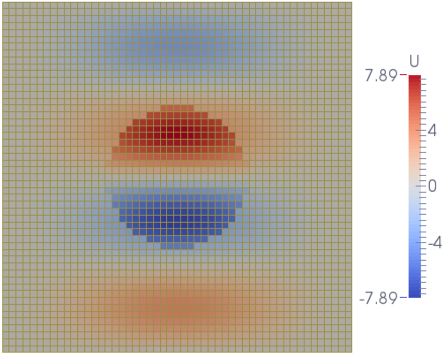
$$\ker \mathbb{L}^h_{\kappa=1,\alpha} \subsetneq \mathcal{E}^{h,\square}_\alpha .$$

Cartesian mesh with $\kappa = 0$:

- Initial condition :



- After 1000 iterations :

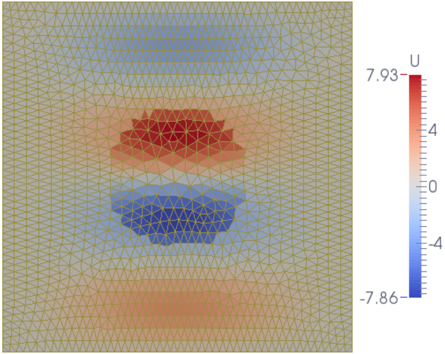


Proposition ($\kappa = 0$ on $\square$)

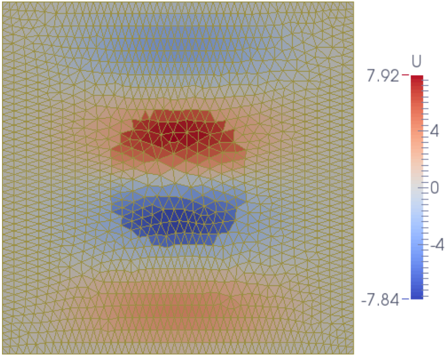
$$\ker\, \mathbb{L}^{h}_{\kappa=0,\alpha} = \mathcal{E}^{h,\square}_{\alpha}.$$

Triangular mesh with $\kappa = 1$:

- Initial condition :



- After 1000 iterations :



Proposition ($\kappa = 1$ on $\triangle$)

$$\ker\, \mathbb{L}^h_{\kappa=1,\alpha} = \mathcal{E}^{h,\triangle}_{\alpha}.$$

Intermediate conclusion : The Godunov scheme

- preserves the incompressible states $q_h \in \mathcal{E}_\alpha^{h,\triangle}$ on **triangular** meshes,
- needs a correction ($\kappa = 0$) to preserve the incompressibles states $q_h \in \mathcal{E}_\alpha^{h,\square}$ on **cartesian** meshes.

Godunov scheme and well-prepared states :

Definition

A state q^0 is said **well-prepared** if

$$\|q^0 - \mathbb{P}_\alpha q^0\|_{L^2_\alpha} = O(M),$$

where $\mathbb{P}_\alpha$ is the orthogonal projection on $\mathcal{E}_\alpha$.

We want to find a correction ($\kappa =?$) on **cartesian** meshes satisfying

$$\|q^0 - \mathbb{P}_\alpha q^0\|_{L^2_\alpha} = O(M) \Rightarrow \forall t \leq ?,\ \|q(t) - \mathbb{P}_\alpha q^0\|_{L^2_\alpha} = O(M)$$

and that allows to recover the Godunov scheme when the Mach number goes to 1.

Initial condition :

- $a_\star = 1$.
- $M = 10^{-4}$.
- $q^0 = q_1^0 + \mathbf{M} q_2^0$ with

$$\begin{cases} r_1^0(x,y) = 1, \\ (\alpha \mathbf{u}_1)^0 = \nabla \times \psi, \end{cases} \Rightarrow q_1^0 \in \mathcal{E}_\alpha$$

  and

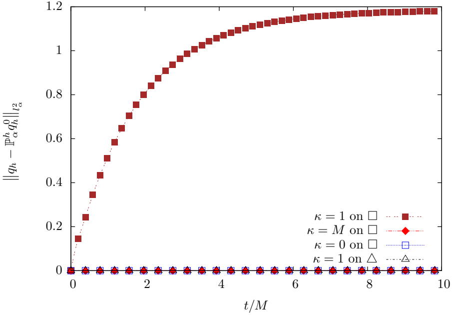
$$\begin{cases} r_2^0(x,y) = 0, \\ \mathbf{u}_2^0 = \nabla \phi, \\ \|q_2^0\|_{L^2_\alpha} = 1, \end{cases} \Rightarrow q_2^0 \in \mathcal{E}_\alpha^\perp$$

  then

$$\|q^0 - \mathbb{P}_\alpha q^0\|_{L^2_\alpha} = \|M q_2^0\|_{L^2_\alpha} = M = O(M).$$

- We will plot $\|q - \mathbb{P}_\alpha q^0\|_{L^2_\alpha}(t)$ as a function of the time.

Cartesian mesh with $\Delta x = \Delta y = 0.02,\ M = 10^{-4}$:

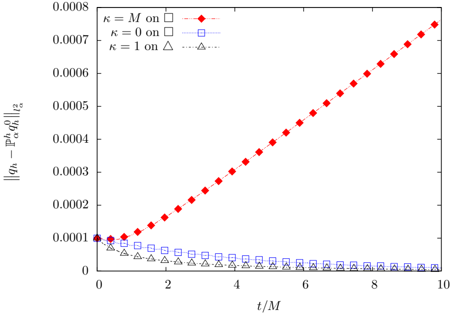


Théorème ($\kappa = 1$ on $\square$)

$$\forall C_1 > 0,\ \exists C_2(C_1) > 0,\ \exists C_3(C_1) > 0,\ \|q^0 - \mathbb{P}_\alpha^{h,\square} q^0\|_{L^2_\alpha} = C_1 M$$
$$\Rightarrow \forall t \geq C_2 M,\ \|q - \mathbb{P}_\alpha^{h,\square} q^0\|_{L^2_\alpha}(t) \geq C_3 \min(\Delta x, \Delta y),$$

for all $M \leq \frac{C_3}{C_1} \min(\Delta x, \Delta y)$.

Cartesian mesh with $\Delta x = \Delta y = 0.02,\ M = 10^{-4}$:

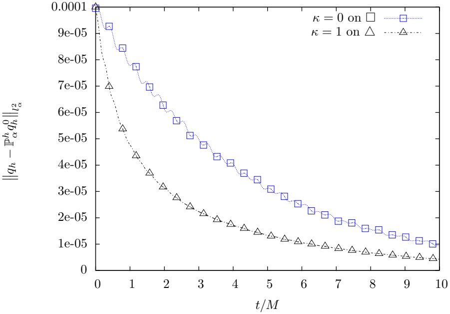


Théorème ($\kappa = M$ on $\square$)

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q^0 - \mathbb{P}_\alpha^{h,\square} q^0\|_{L^2_\alpha} = C_1 M$$
$$\Rightarrow \forall t \in [0; C_2 M],\ \|q - \mathbb{P}_\alpha^{h,\square} q^0\|_{L^2_\alpha}(t) \leq C_3 M,$$

where C_3 does not depend on M.

Cartesian mesh with $\Delta x = \Delta y = 0.02,\ M = 10^{-4}$:



Théorème ($\kappa = 1$ on $\triangle$ and $\kappa = 0$ on $\square$)

$$\forall C_1, C_2 > 0,\ \exists C_3(C_1, C_2) > 0,\ \|q^0 - \mathbb{P}_\alpha^{h,\triangle\ or\ \square} q^0\|_{L^2_\alpha} = C_1 M$$
$$\Rightarrow \forall t \geq 0,\ \|q - \mathbb{P}_\alpha^{h,\triangle\ or\ \square} q^0\|_{L^2_\alpha}(t) \leq C_3 M,$$

where C_3 does not depend on M.

Conclusions et perspectives

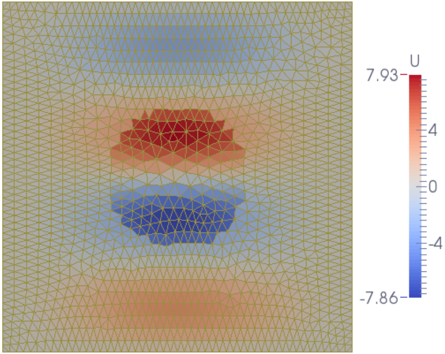
Conclusions :

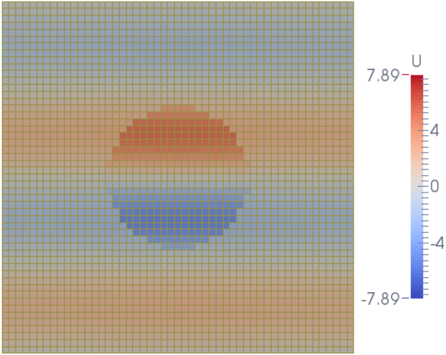
- We explain the influence of cell geometry at low Mach :
  - good behavior of the Godunov scheme ($\kappa = 1$) on **triangular** meshes ($\ker \mathbb{L}^{h,\triangle}_{\kappa=1,\alpha} = \mathcal{E}^{h,\triangle}_{\alpha}$).
  - bad behavior of the Godunov scheme ($\kappa = 1$) on **cartesian** meshes ($\ker \mathbb{L}^{h,\square}_{\kappa=1,\alpha} \subsetneq \mathcal{E}^{h,\square}_{\alpha}$).
- We propose two corrections at low Mach number on **cartesian** meshes :
  - $\kappa = 0$: good correction but it is not continuous against the Mach number.
  - $\kappa = M$: good correction continuous against the Mach number.

Perspectives :

- Test the method in the non-linear case.
- Prove similar properties for the non-linear case.

Thank you for your attention !





S. Dellacherie, P. Omnes. On the Godunov scheme applied to the variable cross-section linear equation. FVCA6, (4) :313–321, 2011.